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Economic Theory

Real indeterminacy of equilibria in a sunspot economy with inside money*

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Summary. In a two-period sunspot economy with inside money and S possible realizations of the sunspot, we prove that, generically in the space of utility functions, there are $S - 1$ degrees of real indeterminacy. This result generalizes the previously known result for sunspot models that, generically in endowments, there is at least one degree of real indeterminacy. The proof involves showing that generically the equilibrium allocation is different across states for some household. This property allows us to perturb the utility function in a simple way and to apply standard transversality arguments to prove our main theorem.

1 Introduction

This paper is an attempt to generalize the real indeterminacy result obtained in the incomplete market model to the sunspot case. In the former framework, if assets pay in units of account, for an open and dense set of endowments, the set of equilibrium allocations exhibits a degree of indeterminacy related to the degree of uncertainty (Balasko and Cass [3]; Geanakoplos and Mas-Colell [5]). As the seminal work by Cass [4] suggests, this result should generalize to the case of extrinsic uncertainty, or sunspots. However, the techniques used in the papers cited do not suffice to verify that claim. This is because they rely on state-dependent perturbations of endowments which cannot be performed when uncertainty is extrinsic, that is when endowments and utility functions are constant across states of the world.

Cass [4] and Siconolfi [10] have shown that equilibrium allocations have, typically, at least one degree of indeterminacy. Using a different approach, Siconolfi and Villanacci [11] have shown that the degree of indeterminacy is related to the degree of uncertainty in the same way as in the case of intrinsic uncertainty, in a model without aggregate risk (i.e., where aggregate, but not individual, endowments are constant across states).

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In this paper, we analyze the standard sunspot model with only inside money and we prove two main results. First, we show that, for a generic set of utility functions, there exists an equilibrium in which there is at least one household whose consumption is different across states. Second, we show that the degree of real indeterminacy is equal to the one obtained in the model with intrinsic uncertainty.

Our work differs from previous papers on sunspots, because we perturb utility functions rather than endowment vectors. This is made possible by employing a technique developed in Kajii [7] parameterizing utility functions through a local parameterization of a finite-dimensional submanifold of the utility function space. This enables us to apply standard transversality arguments to show that, generically in utility functions, households transfer wealth from the first to the second period. This in turn implies that consumption in different states will be different, so that local perturbations of the utility functions induce independent perturbations of the aggregate excess demand function. Therefore the equilibrium set has a manifold structure. Using this result and the equivalency between equilibria in the inside money model and those in a naturally related class of numéraire asset models (see Geanakoplos and Mas-Colell [5]), we then show that for any two vectors of returns of the numéraire asset chosen in a $(S-1)$-dimensional space (where S is the number of states) we get two different equilibria of the inside money economy. Technically speaking, we show that the set of sunspot equilibrium allocations contains the image of a $(S-1)$-dimensional manifold via a C^1, one-to-one function; all the elements in that image are sunspot equilibria. Whether these results can be generalized to an economy with multiple assets is still an open question.

The rest of the paper is organized as follows. Section 2 contains the setup of the model. There we define the space of utility functions we use, along with the topology we impose on it. In Sect. 3, we show the equivalency between equilibria in our model and equilibria in a numéraire asset model. We then prove that, generically in the utility function space, there is some wealth transfer between periods, and hence that an equilibrium with different consumption across states exists. The last part of that section is devoted to establishing the indeterminacy result.

2 The model

We consider a pure exchange economy which lasts for two periods. There are S states of the world in the second period. The superscript $s = 0, 1, \ldots, S$ denotes, for $s = 0$, the first period and, for $s > 0$, one of the S possible states in the second period.

For every $s \geqq 0$, there is a spot market for C physical commodities, labelled by the superscript $c = 1, \ldots, C$. For $s = 0$, there is also a market for inside money. One unit of this asset costs its owner q units of account in $s = 0$ and pays him one unit of account in every $s > 0$.

a. Notation and assumptions

$p^s = (p^{s1}, \ldots, p^{sC}) \in \mathbb{R}^C_{++}$ is the price vector for the C commodities traded in spot s, and $p = (p^0, p^1, \ldots, p^s) \in \mathbb{R}^{C(S+1)}_{++}$. We normalize the price vectors as $p \in P = \{p \in \mathbb{R}^{C(S+1)}_{++} | p^{01} = p^{11} = 1\}$. The price of inside money is denoted by $q \in \mathbb{R}_{++}$. Note that $q \in \mathbb{R}_{++}$ if and only if q does not allow arbitrage possibilities. There are H households labelled by the subscript $h = 1, \ldots, H$. $x_h^s = (x_h^{s1}, \ldots, x_h^{sC}) \in \mathbb{R}_+^C$ and $e_h^s = (e_h^{s1}, \ldots, e_h^{sC}) \in \mathbb{R}_{++}^C$ are the consumption and the endowment vectors of household h in spot s, respectively. We use the notation $x_h = (x_h^0, x_h^1, \ldots, x_h^S)$, $x^s = (x_1^s, \ldots, x_H^s)$ and $x = (x_1, \ldots, x_H)$. The holding of inside money by household h is denoted by $m_h \in \mathbb{R}$.

We assume that:

A1 $H > S$,
A2 Second period endowments are sunspot-invariant: $e_h^s = e_h^1$ for all h and $s > 0$,
A3 Preferences of all households h are represented by an expected utility function:

$$u_h: \mathbb{R}_+^{C(S+1)} \to \mathbb{R}$$

$$u_h(x_h) = \sum_{s=1}^{S} \pi^s v_h^s(x_h^0, x_h^s),$$

where $\pi^S > 0$ for $s \geqq 1$ with $\sum_{s=1}^{S} \pi^s = 1$ are the probabilities of the realization of s and $v_h^s: \mathbb{R}_+^{2C} \to \mathbb{R}$ is, for any h, such that

V1 $v_h^s = v_h$ for any $s > 0$,
V2 v_h is C^2 (on $\mathbb{R}_+^{2C}$),
V3 v_h is differentiably strictly increasing, i.e., $Dv_h(x) \gg 0$ for all $x \in \mathbb{R}_{++}^{2C}$,
V4 v_h is differentiably strictly concave, i.e., $D^2 v_h(x)$ is negative definite for any $x \in \mathbb{R}_{++}^{2C}$, and
V5 For $x^* \in \mathbb{R}_+^{2C} \setminus \{x \in \mathbb{R}_+^{2C} | 0 \leqq x < a\}$, $\{x \in \mathbb{R}_+^{2C} | v_h(x) = v_h(x^*)\}$ is contained in $\mathbb{R}_{++}^{2C}$, where a is a fixed vector satisfying $0 \ll a \ll (e_h^0, e_h^1)$ for $h = 1, \ldots, H$.

Remark 2.1. Assumptions A1, A3, V2, V3 and V4 are standard. Assumptions A2 and V1 guarantee that uncertainty is extrinsic. Assumption V5 insures that the solutions to the households' maximization problems belong to $\mathbb{R}_{++}^{2C}$.

b. Space of economies

Throughout the paper, we fix the endowment vector of all households and consider the space of utility functions as our space of economies. Since the set of feasible allocations is bounded, we can find $w \in \mathbb{R}^{2C}$ such that, for any feasible allocation x, $0 \leqq (x_h^0, x_h^s) < w$ for all s and h. Let $X = \{x \in \mathbb{R}^{2C} | 0 \leqq x \leqq w\}$. We consider the topology of the C^2 uniform convergence on $C^2(X, \mathbb{R})$ and endow $V = \{v: \mathbb{R}_+^{2C} \to \mathbb{R} | v$ satisfies V1 through V5$\}$ with the relative topology from it.[1] Our space of economies is $V^H = \times_{h=1}^{H} V$ endowed with the product topology.[2] We use $E(v)$ to denote the

[1] Observe that $C^2(X, \mathbb{R})$ is metrizable, complete and separable (Mas-Colell [8, p. 50]) and therefore second countable (Munkres [9, p. 194]). Also $C^2(X, \mathbb{R})$ is a Banach space. Moreover V is an open subset of $C^2(X, \mathbb{R})$.

[2] We identify two utility functions that coincide on X. Note that the set of equilibria does not depend on the value of v_h outside of X.

economy identified by $v \in V^H$. In this paper we will say that a property holds "generically" if there is an open and dense subset V^* of V^H such that the property holds for every economy $v \in V^*$.

c. Household behavior

For $(p, q) \in P \times \mathbb{R}_{++}$, every household solves

$$\underset{x_h, m_h}{\text{maximize}} \sum_{s=1}^{S} \pi^s v_h(x_h^0, x_h^s)$$

$$\text{subject to } p^0(x_h^0 - e_h^0) + q m_h = 0,$$
$$p^s(x_h^s - e_h^1) - m_h = 0, \quad s \geqq 1,$$
$$\text{and} \quad x_h \geqq 0.$$

The solution to the problem determines demand functions, $x_h^s(p, q, v_h)$, $s \geqq 0$, and $m_h(p, q, v_h)$, for goods and inside money.

Definition. $(p, q, x, m) \in P \times \mathbb{R}_{++} \times \mathbb{R}_+^{C(S+1)H} \times \mathbb{R}^H$ is a *financial equilibrium* of $E(v)$ if $x_h^s = x_h^s(p, q, v_h)$ and $m_h = m_h(p, q, v_h)$ for all s and h, $\sum_{h=1}^{H} (x_h^0 - e_h^0) = 0$, $\sum_{h=1}^{H} (x_h^s - e_h^1) = 0$, $s \geqq 1$, and $\sum_{h=1}^{H} m_h = 0$. If $x^s \neq x^{s'}$ whenever $s \neq s'$, then it is called a *sunspot equilibrium.*

Our objective is to show that generically there are $S - 1$ degrees of indeterminacy in sunspot equilibria.

3 Indeterminacy of sunspot equilibria

In this section we consider a numéraire asset model M^n with one asset, parameterized by the yield vector y and the households' utility functions v. It is easily shown that for any y, an equilibrium of this numéraire asset model corresponds to an equilibrium of the inside money model. Moreover, we show that, generically in utility functions, the households' holdings of the asset are non-zero at an equilibrium. These results are then used to establish the generic indeterminacy of sunspot equilibria.

A numéraire asset economy is characterized by a vector of asset returns y and by a profile of utility functions $v \in V^H$, and is denoted by $E^n(y, v)$. We restrict $y = (y^1, \ldots, y^S)$ to belong to $Y = \{y \in \mathbb{R}_{++}^S \mid y^1 = 1 \text{ and } y^s \neq y^{s'} \text{ for all } s, s' \geqq 1, s \neq s'\}$, where y^s is the number of units of good 1 delivered in state s. Using "^" to denote variables in this model, the price of the asset in period zero is $\hat{q}$, the set of normalized prices is $\hat{P} = \{\hat{p} \in \mathbb{R}_{++}^{C(S+1)} \mid \hat{p}^{s1} = 1, s \geqq 0\}$ and the set of no-arbitrage asset prices is $\hat{Q} = \mathbb{R}_{++}$.

The household optimization problem is

$$\underset{\hat{x}_h, \hat{b}_h}{\text{maximize}} \sum_{s=1}^{S} \pi^s v_h(\hat{x}_h^0, \hat{x}_h^s)$$

subject to $\hat{p}^0(\hat{x}_h^0 - e_h^0) + \hat{q}\hat{b}_h = 0,$

$\hat{p}^s(\hat{x}_h^s - e_h^1) - y^s\hat{b}_h = 0, \quad s \geqq 1,$

and $\hat{x}_h \geqq 0.$

Household demand functions for goods and asset are $\hat{x}_h^s(\hat{p}, \hat{q}, y, v_h)$, $s \geqq 0$, and $\hat{b}_h(\hat{p}, \hat{q}, y, v_h)$, respectively.

By Walras' law, the market for the first commodity at each spot clears whenever the other markets clear. So we define the following aggregate excess demand function:

$$\hat{\phi}: \hat{P} \times \mathbb{R}_{++} \times Y \times V^H \to \mathbb{R}^{(C-1)(S+1)+1} \text{ such that}$$

$$\hat{\phi}(\hat{p}, \hat{q}, y, v) = \left(\left(\sum_{h=1}^{H} (\hat{x}_h^{0c}(\hat{p}, \hat{q}, y, v_h) - e_h^{0c}) \right)_{c=2}^{C}, \left(\sum_{h=1}^{H} (\hat{x}_h^{sc}(\hat{p}, \hat{q}, y, v_h) - e_h^{1c}) \right)_{s=1, c=2}^{S,C}, \sum_{h=1}^{H} \hat{b}_h(\hat{p}, \hat{q}, y, v_h) \right).$$

Definition. $(\hat{p}, \hat{q}, \hat{x}, \hat{b}) \in \hat{P} \times \mathbb{R}_{++} \times \mathbb{R}_{+}^{C(S+1)H} \times \mathbb{R}^H$ is a *financial equilibrium* of $E^n(y, v)$ if $\hat{x}_h^s = \hat{x}_h^s(\hat{p}, \hat{q}, y, v_h)$ and $\hat{b}_h = \hat{b}_h(\hat{p}, \hat{q}, y, v_h)$ for all s and h, $\sum_{h=1}^{H} (\hat{x}_h^0 - e_h^0) = 0$, $\sum_{h=1}^{H} (\hat{x}_h^s - e_h^1) = 0$, $s \geqq 1$, and $\sum_{h=1}^{H} \hat{b}_h = 0$, i.e., $\hat{\phi}(\hat{p}, \hat{q}, y, v) = 0$.

The following lemma states the relation between the equilibria of $E^n(y, v)$ and those of $E(v)$:

Lemma 1. *If* $(\hat{p}, \hat{q}, \hat{x}, \hat{b})$ *is a financial equilibrium of* $E^n(y, v)$, *then* $(p, q, x, m) = \left(\left(\hat{p}^0, \frac{\hat{p}^1}{y^1}, \ldots, \frac{\hat{p}^S}{y^S} \right), \hat{q}, \hat{x}, \hat{b} \right)$ *is a financial equilibrium of* $E(v)$.

Proof of Lemma 1. Obvious. □

Remark 3.1. The two equilibria in Lemma 1 have the same commodity allocations.

Lemma 2. *Let* $(\hat{p}, \hat{q}, \hat{x}, \hat{b}) \in \hat{P} \times \mathbb{R}_{++} \times \mathbb{R}_{+}^{C(S+1)H} \times \mathbb{R}^H$ *be a financial equilibrium of* $E^n(y, v)$. *Then, if* $\hat{b}_h \neq 0$, $\hat{x}_h^s \neq \hat{x}_h^{s'}$ *for all* $s, s' \geqq 1$, $s \neq s'$.

Proof of Lemma 2.

Case 1. $\hat{p}^s = \hat{p}^{s'}$. From the budget constraints, we have $\hat{p}^s(\hat{x}_h^s - e_h^1) = y^s\hat{b}_h$ and $\hat{p}^{s'}(\hat{x}_h^{s'} - e_h^1) = y^{s'}\hat{b}_h$. Since $y^s \neq y^{s'}$ and $\hat{b}_h \neq 0$, we conclude $\hat{x}_h^s \neq \hat{x}_h^{s'}$.

Case 2. $\hat{p}^s \neq \hat{p}^{s'}$. From the first order conditions, we have

$D_1 v_h(\hat{x}_h^0, \hat{x}_h^s) = \frac{\hat{\lambda}_h^s}{\pi^s}\hat{p}^s$ and $D_1 v_h(\hat{x}_h^0, \hat{x}_h^{s'}) = \frac{\hat{\lambda}_h^{s'}}{\pi^{s'}}\hat{p}^{s'}$. Since $\hat{p}^{s1} = \hat{p}^{s'1} = 1$, $\hat{x}_h^s \neq \hat{x}_h^{s'}$ (otherwise $\hat{p}^s = \hat{p}^{s'}$, a contradiction). □

Theorem 1. *There exists an open and dense subset* V^* *of* V^H *such that for all* $(\hat{p}, \hat{q}, y, v) \in \hat{P} \times \mathbb{R}_{++} \times Y \times V^*$, $\hat{\phi}(\hat{p}, \hat{q}, y, v) = 0$ *implies* $\hat{b}_h(\hat{p}, \hat{q}, y, v_h) \neq 0$ *for some* $h \in H$.

The proof of Theorem 1 is deferred to the Appendix.

Corollary to Theorem 1. *Generically a sunspot equilibrium exists.*

Proof of the corollary. Consider V^* obtained in Theorem 1. Then for any $v \in V^*$ and $y \in Y$ there exists $(\hat{p}, \hat{q}) \in \hat{P} \times \mathbb{R}_{++}$ such that $\hat{\phi}(\hat{p}, \hat{q}, y, v) = 0$. (Geanakoplos and Polemarchakis [6]). Then by Theorem 1, $\hat{b}_h(\hat{p}, \hat{q}, y, v_h) \neq 0$ for some $h \in H$. By Lemma 2, $\hat{x}_h^s \neq \hat{x}_h^{s'}$ for all $s, s' \geqq 1$, $s \neq s'$. So if we construct a financial equilibrium of $E(v)$ as in Lemma 1, we get a sunspot equilibrium. □

We are now to show that the equilibrium of our original economy (with inside money) is indeterminate.

Theorem 2. *There exists an open and dense subset V^{**} of V^H such that for all $v \in V^{**}$ the set of sunspot equilibrium allocations of $E(v)$ contains the image of an $(S-1)$-dimensional manifold by a C^1, one-to-one function.*

Proof of Theorem 2. We first prove that 0 is a regular value of $\hat{\phi}$,

$$\hat{\phi}: \hat{P} \times \mathbb{R}_{++} \times Y \times V^* \to \mathbb{R}^{(C-1)(S+1)+1}.$$

Let $(\hat{p}, \hat{q}, y, v) \in \hat{P} \times \mathbb{R}_{++} \times Y \times V^*$. Then $\hat{\phi}(\hat{p}, \hat{q}, y, v) = 0$ implies by Theorem 1 and Lemma 2 that there is a household h such that $\hat{x}_h^s \neq \hat{x}_h^{s'}$ for all $s, s' \geqq 1$, $s \neq s'$. Hence we can essentially repeat the argument made in the proof of Theorem 1. That is, we can perturb this household's utility function to prove that $D_v\hat{\phi}(\hat{p}, \hat{q}, y, v)$ is onto, i.e., 0 is a regular value of $\hat{\phi}$. Hence $\hat{\phi}^{-1}(\{0\})$ is a smooth manifold (Corollary 17.2, Abraham and Robbin [2]).

Now let $\beta: \hat{\phi}^{-1}(\{0\}) \to Y \times V^*$ be a projection and $V^{**} \equiv \{v \in V^* \mid \exists y \in Y$ such that (y, v) is a regular value of $\beta\}$. We shall show that (i) for any $v \in V^{**}$ the set of equilibrium allocations of $E(v)$ contains the image of a $(S-1)$-dimensional manifold via a C^1, one-to-one function, and (ii) V^{**} is open and dense in V^*.

(i) Consider $\hat{v} \in V^{**}$ and take $\hat{y} \in Y$ where $(\hat{y}, \hat{v})$ is a regular value of β. Then there exists $(\hat{p}, \hat{q}) \in \hat{P} \times \mathbb{R}_{++}$ such that $\hat{\phi}(\hat{p}, \hat{q}, \hat{y}, \hat{v}) = 0$ (Geanakoplos and Polemarchakis [6]). Hence by the implicit function theorem, there exists an open neighborhood $\hat{F} \subset Y \times V^*$ of $(\hat{y}, \hat{v})$ and a C^1 function $\gamma: \hat{F} \to \hat{P} \times \mathbb{R}_{++}$ such that $\gamma(\hat{y}, \hat{v}) = (\hat{p}, \hat{q})$ and $\hat{\phi}(\gamma(y, v), y, v) = 0$ for all $(y, v) \in \hat{F}$. Now construct a function,

$$\delta: F \equiv \{y \in Y \mid (y, \hat{v}) \in \hat{F}\} \to P \times \mathbb{R}_{++} \quad \text{such that} \quad y \mapsto \left(\hat{p}^0, \left(\frac{\hat{p}^s}{y^s}\right)_{s=1}^S, \hat{q}\right).$$

δ is C^1 and F is open in Y (being the section of $\hat{F}$ at $\hat{v}$). Define a function,

$$\mu: F \to \mathbb{R}^{(S+1)CH} \quad \text{such that} \quad y \mapsto (x_h(\delta(y), y, \hat{v}_h))_{h=1}^H.$$

μ is C^1. We now check that μ is one-to-one. Let $y, y' \in F$, $y \neq y'$, and suppose $\mu(y) = \mu(y')$. Then $\hat{p} = \hat{p}'$ where $\hat{p}$ and $\hat{p}'$ are obtained by $\gamma(y, \hat{v})$ and $\gamma(y', \hat{v})$. By the budget constraints of the economies $E^n(y, \hat{v})$ and $E^n(y', \hat{v})$ we have $y^s\hat{b}_h = y'^s\hat{b}'_h$ for all s and h, so that for $s = 1$, $\hat{b}_h = \hat{b}'_h$ for any h. But this is a contradiction, since by Theorem 1, we know that there is some h such that $\hat{b}_h \neq 0$.

(ii) By Smale's Density Theorem (Abraham and Robbin [2]), the set $A \equiv \{(y, \hat{v}) \in Y \times V^* \mid (y, \hat{v})$ is a regular value of $\beta\}$ is dense in $Y \times V^*$. Hence V^{**}, its projection into V^*, is dense as well.

Let C be the set of critical points of β. $A = Y \times V^* \backslash \beta(C)$. Now C is closed (Abraham et al. [1, p. 556]) and β is proper (by a standard argument), hence $\beta(C)$ is closed and A is open in $Y \times V^*$. Therefore V^{**} is open in V^*. So V^{**} is open and dense in V^* and hence in V^H. □

Appendix

Our objective is to prove Theorem 1. To begin with, we need to define a numéraire asset model M_k^n with k states in the second period. Variables in this model are augmented by "$\tilde{}$". The consumption set is $\mathbb{R}_+^{C(k+1)}$, the return structure is $[-\tilde{q}, y^1, y^2, \ldots, y^k]^T \equiv [-\tilde{q}, y]^T \in \mathbb{R}^{k+1}$, and the set of normalized prices is $\tilde{P}_k = \{\tilde{p} \in \mathbb{R}_{++}^{C(k+1)} | \tilde{p}^{s1} = 1, s = 0, \ldots, k\}$. In this appendix we normalize the asset prices and yields as follows:

$$\tilde{A}_k \equiv \{(\tilde{q}, y) \in \mathbb{R}^{k+1} | \|(\tilde{q}, y)\| = 1 \text{ and there is no } b \in \mathbb{R}$$
$$\text{such that } [-\tilde{q}, y^1, y^2, \ldots, y^k]^T b > 0\}.$$

Households' maximization problems, demand functions and market clearing conditions are defined in a way consistent with the nature of the model M_k^n. We denote the economy with yield structure y and utility functions v by $E_k^n(y, v)$. Moreover define a function,

$$\tilde{\phi}: \tilde{P}_k \times \tilde{A}_k \times V^H \to \mathbb{R}^{(C-1)(k+1)+1},$$
$$\tilde{\phi}(\tilde{p}, \tilde{q}, y, v) = (\tilde{\phi}_c(\tilde{p}, \tilde{q}, y, v), \tilde{\phi}_b(\tilde{p}, \tilde{q}, y, v)).$$

Definition. $(\tilde{p}, \tilde{q}, \tilde{x}, \tilde{b}) \in \tilde{P}_k \times \mathbb{R} \times \mathbb{R}_+^{C(k+1)H} \times \mathbb{R}^H$ is a *financial equilibrium* of $E_k^n(y, v)$ if $\tilde{\phi}(\tilde{p}, \tilde{q}, y, v) = 0$.

Lemma 3. *Let* $k = 1, \ldots, S$ and $(\tilde{p}, \tilde{q}, \tilde{x}, \tilde{b}) \in \tilde{P}_k \times \mathbb{R} \times \mathbb{R}_+^{C(k+1)H} \times \mathbb{R}^H$ *be a financial equilibrium of* $E_k^n(y, v)$. *Then*
(a) *If* $\tilde{b}_h = 0$ *and* $\tilde{p}^s = \tilde{p}^{s'}$, *then* $\tilde{x}_h^s = \tilde{x}_h^{s'}$.
(b) *If* $\tilde{x}_h^s = \tilde{x}_h^{s'}$, *then* $\tilde{p}^s = \tilde{p}^{s'}$.

Proof of Lemma 3. Straightforward. □

Lemma 4. *Consider the model* M_1^n. *There exists an open and dense subset* V_1^* *of* V^H *such that for all* $(\tilde{p}, \tilde{q}, y, v) \in \tilde{P}_1 \times \tilde{A}_1 \times V_1^*$, $\tilde{\phi}(\tilde{p}, \tilde{q}, y, v) = 0$ *implies* $\tilde{b}_h(\tilde{p}, \tilde{q}, y, v_h) \neq 0$ *for some* $h \in H$.

Proof of Lemma 4.[3] Let $(\tilde{p}, \tilde{q}, y, v) \in \tilde{P}_1 \times \tilde{A}_1 \times V^H$ satisfy

$$\tilde{\phi}_c(\tilde{p}, \tilde{q}, y, v) = 0$$
$$\tilde{b}_1(\tilde{p}, \tilde{q}, y, v_1) = 0$$
$$\vdots$$
$$\tilde{b}_H(\tilde{p}, \tilde{q}, y, v_H) = 0$$

[3] We follow the perturbation in Kajii [7].

and $\tilde{\psi} \equiv (\tilde{\phi}_c, \tilde{b}_1, \ldots, \tilde{b}_H): \tilde{P}_1 \times \tilde{A}_1 \times V^H \to \mathbb{R}^{2(C-1)+H}$. We shall show that $D_v\tilde{\psi}$ evaluated at $(\tilde{p}, \tilde{q}, y, v)$ has rank $2(C-1)+H$.

For $h \in H$, let B_h and G_h be open sets of $\mathbb{R}^{2C}$ such that $\tilde{x}_h(\tilde{p}, \tilde{q}, y, v_h) \in B_h \subset \bar{B}_h \subset G_h \subset X$ and ρ_h be a C^∞ function from $\mathbb{R}^{2C}$ to $\mathbb{R}$ such that $\rho_h(x) = 0$ on B_h and $\rho_h(x) = 1$ on the complement of G_h. For $w_h = (w_h^0, w_h^1) \in \mathbb{R}^{2C}$, define a function $\zeta_h(w_h)$ from $\mathbb{R}^{2C}_+$ to $\mathbb{R}$ by $\zeta_h(w_h)(x) = \rho_h(x)v_h(x) + (1 - \rho_h(x))v_h(x-w)$. Then we can find an open neighborhood $W_h = W_h^0 \times W_h^1$ of $0 \in \mathbb{R}^{2C}$ such that

$$\zeta_h(w_h) \in V \quad \text{for all } w_h \in W_h, \text{ for } h \in H,$$

i.e., $\zeta = (\zeta_1, \ldots, \zeta_H)$ is a smooth parameterization of an $2CH$-dimensional submanifold of V^H about $v = (v_1, \ldots, v_H)$. Using this parameterization, we can naturally regard $\tilde{\psi}(\tilde{p}, \tilde{q}, y; \cdot)$ as a function from $W = W_1 \times \cdots \times W_H$ to $\mathbb{R}^{2(C-1)+H}$ and write $\tilde{\psi}(\tilde{p}, \tilde{q}, y; w)$. If the rank of $D_w\tilde{\psi}$ is $2(C-1)+H$, then so is that of $D_v\tilde{\psi}$.

Now $D_x\zeta_h(w_h)(x) = Dv_h(x-w)$ for x sufficiently close to $\tilde{x}_h(\tilde{p}, \tilde{q}, y, v_h)$. Let $w_1^0 \in W_1^0$ be such that $\tilde{p}^0 w_1^0 = 0$ and set $w = ((w_1^0, 0), 0, \ldots, 0) \in W$. Then $(\tilde{x}'_1, \tilde{b}'_1)$ defined by $(\tilde{x}_1^0)' = \tilde{x}_1^0 + w_1^0$, $(\tilde{x}_1^1)' = \tilde{x}_1^1$, and $\tilde{b}'_1 = \tilde{b}_1$ solves household 1's maximization problem when her utility function is $\zeta_1(w)$. That is, if we perturb ζ by w, $\tilde{\psi}$ changes by $((\bar{w}_1^0, 0), 0, \ldots, 0) \in \mathbb{R}^{2(C-1)+H}$, where $\bar{w}_1^0$ is the vector composed by the last $C-1$ elements of w_1^0.

Similarly, for $w_1^1 \in W_1^1$ such that $\tilde{p}^1 w_1^1 = 0$, and $w' = ((0, w_1^1), 0, \ldots, 0) \in W$, the perturbation of ζ by w' changes $\tilde{\psi}$ by $((0, \bar{w}_1^1), 0, \ldots, 0) \in \mathbb{R}^{2(C-1)+H}$.

Moreover for α small enough such that $w_h^0 = (-\alpha q, 0, \ldots, 0) \in \mathbb{R}^C$, $w_h^1 = (\alpha, 0, \ldots, 0)$, and $w'' = (0, \ldots, 0, w_h, 0, \ldots, 0) \in W$, the perturbation of ζ by w'' changes $\tilde{\psi}$ by $((0,0), 0, \ldots, 0, \alpha, 0, \ldots, 0)$ (See Fig. 1).

Thus, after a suitable coordinate change,

$$D_w\tilde{\psi} = \begin{pmatrix} I_{2(C-1)\times 2(C-1)} & * & * \\ 0 & I_{H\times H} & * \end{pmatrix}.$$

Therefore we have shown that $D_v\tilde{\psi}$ evaluated at $(\tilde{p}, \tilde{q}, y, v)$ has rank $2(C-1)+H$. By the transversality theorem (Abraham and Robbin [2]), there exists a dense subset

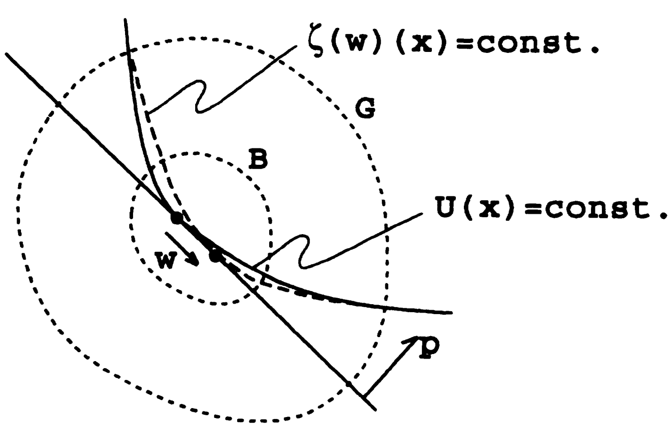

Fig. 1

V_1^* of V^H such that for all $v \in V_1^*$, 0 is a regular value of $\tilde{\psi}_v = \tilde{\psi}(\cdot,\cdot,\cdot;v)$. But since the domain has lower dimension than the range, there is no solution to $\tilde{\psi}_v = 0$.

Finally we shall show that V_1^* is open in V^H. Without loss of generality we assume

$$V_1^* = \{v \in V^H | 0 \text{ is a regular value of } \tilde{\psi}_v\} = V^H \backslash \chi(\tilde{\psi}^{-1}(\{0\})),$$

where χ is a projection of $\tilde{\psi}^{-1}(\{0\})$ into V^H.

Since $\tilde{\psi}$ is continuous, $\tilde{\psi}^{-1}(\{0\})$ is closed. A standard argument shows that χ is proper[4] and hence V_1^* is open. □

For $k = 1, \ldots, S$ we consider the following set of lemmas (where $V_0^* \equiv V^H$):

Lemma 4(*k*). *Consider the model M_k^n. There exists an open and dense subset V_k^* of V_{k-1}^* such that for all $(\tilde{p}, \tilde{q}, y, v) \in \tilde{P}_k \times \tilde{A}_k \times V_k^*$, $\tilde{\phi}(\tilde{p}, \tilde{q}, y, v) = 0$ implies $\tilde{b}_h(\tilde{p}, \tilde{q}, y, v_h) \neq 0$ for some $h \in H$.*

Lemma 5(*k*). *There exists an open and dense subset V_k^* of V_{k-1}^* such that for all M_j^n, $j = 1, \ldots, S$ and for all $(\tilde{p}, \tilde{q}, y, v) \in \tilde{P}_j \times \tilde{A}_j \times V_k^*$,*
(i) $\tilde{\phi}(\tilde{p}, \tilde{q}, y, v) = 0$,
(ii) $\tilde{b}_h(\tilde{p}, \tilde{q}, y, v_h) = 0$, for all $h \in H$, and
(iii) $\exists (p^1, \ldots, p^k) \in \mathbb{R}_{++}^{Ck}$ such that for all $s = 1, \ldots, j$, $\tilde{p}^s = p^i$ for some $i = 1, \ldots, k$,
cannot hold.

Notice that Lemma 4(1) is precisely Lemma 4 that we proved above and from Lemma 4(S) we can easily prove Theorem 1.[5] So it is sufficient to show the following two steps:

(a) $\left\langle \begin{matrix} \text{Lemma } 4(1), \ldots, \text{Lemma } 4(k) \\ \text{Lemma } 5(1), \ldots, \text{Lemma } 5(k-1) \end{matrix} \right\rangle \Rightarrow \text{Lemma } 5(k), k = 1, \ldots, S,$

(Lemma 4(1) ⇒ Lemma 5(1) for $k = 1$)
(b) Lemma 5($k-1$) ⇒ Lemma 4(k), $k = 2, \ldots, S$.

Proof of Step (a). Rely on using an equilibrium $(\tilde{p}, \tilde{q}, \tilde{x}, 0)$ of $E_j^n(y, v)$ to construct an equilibrium of $E_k^n(\bar{y}, u)$ for which (i), (ii) and (iii) hold, leading to a contradiction of Lemma 4(k).[6] □

Proof of Step (b). Let $(\tilde{p}, \tilde{q}, y, v) \in \tilde{P}_k \times \tilde{A}_k \times V_k^*$ satisfy

$$\tilde{\phi}_C(\tilde{p}, \tilde{q}, y, v) = 0$$

[4] See Suda et al. [12] for the detail.

[5] Note that $M_S^n = M^n$ with a necessary change of the normalization. Also, since V_S^* is open and dense in V_{S-1}^*, which is open and dense in V_{S-2}^*, which is...open and dense in V^H, V_S^* is open and dense in V^H.

[6] See Suda et al. [12] for the detail.

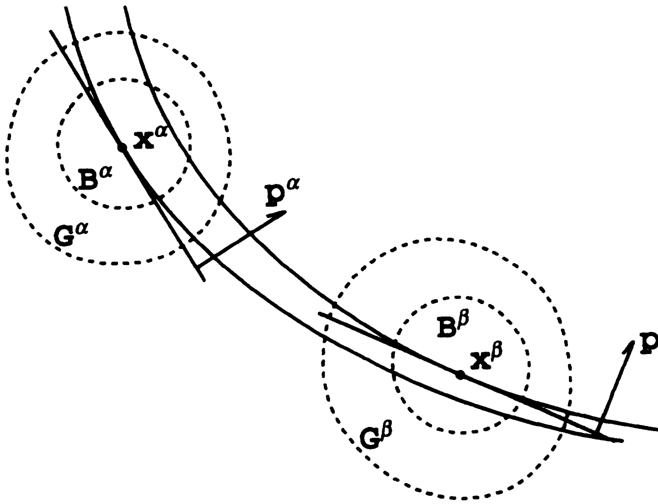

Fig. 2

$$\tilde{b}_1(\tilde{p}, \tilde{q}, y, v_1) = 0$$
$$\vdots$$
$$\tilde{b}_H(\tilde{p}, \tilde{q}, y, v_H) = 0$$

and $\tilde{\psi} \equiv (\tilde{\phi}_C, \tilde{b}_1, \dots, \tilde{b}_H): \tilde{P}_k \times \tilde{A}_k \times V_k^* \to \mathbb{R}^{(C-1)(k+1)+H}$.

Now since $v \in V_k^* \subset V_{k-1}^*$, by Lemma 5($k-1$), $\tilde{p}^s \neq \tilde{p}^{s'}$ for $s \neq s'$, $s, s' \geqq 1$. Thus by Lemma 3(a) and (b), $\tilde{x}_h^s \neq \tilde{x}_h^{s'}$ for all $s \neq s'$, $s, s' \geqq 1$, all $h \in H$. Therefore for $h \in H$ and for $s \geqq 1$, we can find open subsets, B_h^s and G_h^s, of $\mathbb{R}^{2C}$ such that $(\tilde{x}_h^0, \tilde{x}_h^s) \in B_h^s \subset \bar{B}_h^s \subset G_h^s \subset X$ and $G_h^s \cap G_h^{s'} = \varnothing$ for $s \neq s'$. Let ρ_h^s be a C^∞ function from $\mathbb{R}^{2C}$ to $\mathbb{R}$ such that $\rho_h^s(x) = 0$ on B_h^s and $\rho_h^s(x) = 1$ on the complement of G_h^s. For $w_h = (w_h^0, w_h^1, \dots, w_h^k) \in \mathbb{R}^{(k+1)C}$ define a function $\zeta_h(w_h)$ from $\mathbb{R}_+^{2C}$ to $\mathbb{R}$ by

$$\zeta_h(w_h)(x) = (\rho_h^1(x) \cdots \rho_h^k(x)) v_h(x) + \sum_{s=1}^{k} (1 - \rho_h^s(x)) v_h(x - (w_h^0, w_h^s)).$$

Then we can find an open neighborhood $W_h = W_h^0 \times W_h^1 \times \cdots \times W_h^k$ of $0 \in \mathbb{R}^{(k+1)C}$, $h \in H$, such that $(\zeta_1(w_1), \dots, \zeta_H(w_H)) \in V_{k-1}^*$ for all $(w_1, \dots, w_H) \in W_1 \times \cdots \times W_H$. Using this parameterization, we can show by a similar argument as in Lemma 4 that there exists an open and dense subset V_k^* of V_{k-1}^* such that there is no solution to $\tilde{\psi}_v = 0$ for any $v \in V_k^*$. (See Fig. 2 for a general idea of our perturbation.) □

References

1. Abraham, R., Marsden, J.E., Ratiu, T.: Manifolds, tensor analysis, and applications. Reading: Addison-Wesley 1983
2. Abraham, R., Robbin, J.: Transversal mappings and flows. New York: Benjamin 1967
3. Balasko, Y., Cass, D.: The structure of financial equilibrium with exogenous yields: the case of incomplete markets. Econometrica 57, 135–162 (1989)

4. Cass, D.: Sunspots and incomplete financial markets: the leading example. In: Feiwel, G. (ed.) The economics of imperfect competition and employment: Joan Robinson and beyond. London: MacMillan 1989
5. Geanakoplos, J., Mas-Colell, A.: Real indeterminacy with financial assets. J. Econ. Theory **47**, 22–38 (1989)
6. Geanakoplos, J., Polemarchakis, H.: Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete. In: Heller, W.P., Starr, R.M., Starrett, D.A. (eds.) Uncertainty, information and communication: essays in honor of Kenneth J. Arrow, vol. III. Cambridge: Cambridge University Press 1986
7. Kajii, A.: On the role of options in sunspot equilibria. CARESS Working Paper #91-24, University of Pennsylvania, 1991
8. Mas-Colell, A.: The theory of general economic equilibrium: a differentiable approach. Cambridge: Cambridge University Press 1985
9. Munkres, J.R.: Topology: a first course. Englewood Cliffs: Prentice-Hall 1975
10. Siconolfi, P.: Sunspot equilibria and incomplete financial markets. J. Math. Econ. **20**, 327–339 (1991)
11. Siconolfi, P., Villanacci, A.: Real indeterminacy in incomplete financial market economies without aggregate risk. Econ. Theory **1**, 265–276 (1991)
12. Suda, S., Tallon, J.-M., Villanacci, A.: Real indeterminacy of equilibria in a sunspot economy with inside money. CARESS Working Paper #91-15, University of Pennsylvania, 1991