Sharing Model Uncertainty

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Abstract

This paper examines efficient allocations in economies where consumers exhibit heterogeneous smooth ambiguity preferences and face model uncertainty with a common set of identifiable models. Aggregate endowment is ambiguous. We characterize economies where the representative consumer is of the smooth ambiguity type and derive efficient sharing rules. Heterogeneous ambiguity aversion leads to sharing rules that systematically differ from those in vNM-economies. The representative consumer's ambiguity aversion differs from that of the typical consumer; this leads to more compelling asset-pricing predictions. We focus on point-identified models but show that our insights extend to partially-identified models.

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Uncertainty, as opposed to risk, remains a fundamental challenge in decision-making across various economic contexts. Whether in financial markets during the 2007–2009 crisis, policymakers confronting an emerging virus, or farmers grappling with climate change, decision-makers frequently encounter uncertainties that cannot be easily quantified probabilistically. In such settings, understanding how economic institutions may accommodate and potentially hedge against uncertainty becomes crucial.

A key example in macro-finance is the modeling of economic fluctuations as regime-switching processes between "booms" and "busts".[1] While the growth distribution for each regime may be estimated with reasonable accuracy, consumers remain uncertain about which regime will prevail in the future. Suppose consumers perceive this uncertainty as ambiguous and adopt robust strategies. Their choices among contingent consumption plans depend on their level of ambiguity aversion. This raises fundamental questions: What would the efficient allocations be in such circumstances? How does heterogeneous ambiguity aversion impact uncertainty-sharing compared to expected utility settings? What are the implications for asset pricing, particularly the market price of uncertainty and the shape of the pricing kernel? The literature has left these questions largely open, and this paper seeks to fill that gap.

To analyze these issues, we formalize uncertainty as model uncertainty, where a model —characterized by specific parameters and mechanisms— generates a probabilistic forecast of economic outcomes. Following the empirical literature, we assume that these parameters and mechanisms can be *identified* and possibly estimated from objective data. The framework we employ, of model uncertainty with identifiable models where consumers have *smooth ambiguity* preferences (Klibanoff, Marinacci & Mukerji 2005), was incorpo-

[1]We will use this as a running example to illustrate the concepts we introduce in the subsequent analyses. In the context of ambiguity, see (Ju & Miao 2012).

rated into decision-making under ambiguity by (Denti & Pomatto 2022). In the macro-finance stochastic environment referred to earlier, we think of each regime as a model, implying a specific probability distribution on (aggregate) endowment, and the uncertainty about the impending regime as model uncertainty. Experts (e.g., NBER) decide whether the economy was in a recession, based on observations of variables from different sectors of the economy. This announcement is itself an event that identifies the regime.

We investigate efficient allocations in an economy with heterogeneously ambiguity averse consumers where aggregate endowment is ambiguous, that is, multiple identifiable models generate different endowment distributions. Our analysis yields several novel results.We show that while sharing rules align with those in von Neumann–Morgenstern (vNM) economies when the aggregate endowment is unambiguous, they differ systematically when it is ambiguous. For instance, efficient allocations need not be comonotonic with the endowment. A generalized Borch rule is identified to characterize efficient allocations, demonstrating how the assumption of identifiable models adds significant structure on these allocations so as to accommodate consumers' desire to share the uncertainty *about models*.

We characterize and further analyze those economies that admit a smooth ambiguity representative consumer. The characterizing condition, that consumers' utility functions exhibit linear risk tolerance (viz. HARA functions) with common marginal risk tolerance, is precisely the one that characterizes vNM economies where efficient risk sharing rules are linear. Hence, the permissible class of utilities coincides with the standard domain for macro-finance studies. In this setting, efficient allocations are such that all consumers rank models the same way, a property we call "Expected-utility comonotonicity". We pin down the efficient sharing rules and make explicit their systematic departure from those obtained under expected utility.

We also obtain an analytically tractable formulation of the representative consumer, in particular its ambiguity aversion, and establish how that ambiguity aversion relates to the ambiguity aversion of individual consumers in the economy. This, in turn, simplifies the exercise of finding asset pricing implications, and how they depend on the heterogeneity of ambiguity aversion among the consumers. The results obtained about the representative consumer and efficient sharing rules, initially derived assuming that the set of models are *point-identifiable*, in the usual statistical sense, are shown to extend to the case where models are only *set (or, partially)-identifiable.*[2] Set-identifiability is, arguably, a minimum condition that empirically founded models may be expected to satisfy.

The sharing rule we derive implies that the representative consumer would not typically have constant relative ambiguity aversion, as is widely assumed in the macro-finance literature. For instance, even if individual consumers have *constant* relative ambiguity aversion, as long as its level is heterogeneous in the population, the representative consumer will have *decreasing* relative ambiguity aversion. If booms, compared to busts, are associated with better distributions of aggregate output (say, in the sense of first-order stochastic dominance), then the relative ambiguity aversion of the representative consumer is counter-cyclical.

Under heterogeneous ambiguity aversion, the pricing kernel deviates from standard models, providing a potential explanation for empirical puzzles in asset pricing. For example, in a Gaussian environment where uncertainty is primarily about the mean growth rate of the economy, we show that the presence of heterogeneous ambiguity aversion leads to a more volatile and counter-cyclical Sharpe ratio. As (Lettau & Ludvigson 2010) point out, these features are prominent in the data but existing theories struggle to explain these patterns.

[2]That is, the recovery, for each model, is only up to a set of probability laws.

Related literature. The classic theory of efficient risk-sharing in expected utility economies dates back to (Borch 1962), further refined for the HARA class of utility functions by (Wilson 1968), (Cass & Stiglitz 1970) and (Hara, Huang & Kuzmics 2007) among others. (Chateauneuf, Dana & Tallon 2000) extended the comonotonicity result obtained under expected utility to Choquet expected utility with common capacity. (Billot et al. 2000), (Rigotti, Shannon & Strzalecki 2008) and (Ghirardato & Siniscalchi 2018) further studied the case in which aggregate endowment is deterministic and preferences are more general than Choquet-expected-utility preferences (including, for the two latter references, the smooth ambiguity model). (Strzalecki & Werner 2011) and (De Castro & Chateauneuf 2011) characterized properties of efficient risk-sharing when the aggregate endowment is risky but not ambiguous. (Beißner & Werner 2023) extends some of these results to cases where agents have possibly heterogeneous, non-convex ambiguity sensitive preferences. Assuming Maxmin-Expected-Utility (MEU) decision makers à la (Gilboa & Schmeidler 1989), (Wakai 2007) proves that, under HARA with common risk tolerance, efficient allocations are comonotone. To the best of our knowledge, no paper has studied risk-sharing with *ambiguous* aggregate endowments and *heterogeneous* ambiguity aversion. As we find, it is these two ingredients that create and drive differences between risk-sharing in vNM-economies and that in economies with ambiguity aversion.

The remainder of the paper is structured as follows. Section 1 presents our formal model and derive properties of efficient allocations. Section 2 characterizes the economies in which a smooth ambiguity representative consumer exists and derives explicit efficient sharing rules for these economies. Section 3 explores the asset pricing implications of our framework. Proofs are gathered in the Appendix, while additional results and technical details are provided in an Online Appendix.

1 Setting and preliminary results

1.1 Uncertainty and identifiable models

We analyze a pure exchange economy with a finite state space, Ω, where uncertainty includes model uncertainty. A model is defined as a data-generating process, specifying a probability distribution over Ω. The set of possible models is denoted by $\mathcal{P}$, with a generic element $\mathbf{P}$. We require our models to be identifiable: it is possible to infer the true values of a model's underlying parameters after observations are made. Model uncertainty then describes the idea that consumers are missing the information (eventually observable events) that would allow them to identify a probabilistic model.

Example 1. *A ball is drawn from an urn containing 60 red, blue, or yellow balls, in an unknown proportion. The outcome of the draw is the ball's color* $c \in \{r, b, y\}$*. Unlike the classic Ellsberg experiment, the urn's composition is revealed after the draw. It is denoted by* $\gamma \in \Gamma = \{(\rho, \beta, \nu) \in \mathbb{N}^3 : \rho + \beta + \nu = 60\}$*. The composition of the urn is the model uncertainty of this experiment; it can be seen as the lacking probabilistic information that is revealed. A state of the world is thus given by* $\omega = (c, \gamma)$ *revealing* both *the color of the ball drawn* c and *the composition of the urn* γ*; the state space is* $\Omega = \{r, b, y\} \times \Gamma$*. The set of models is* $\mathcal{P} = \{\mathbf{P}_\gamma, \gamma \in \Gamma\}$*, indexed by the composition* γ*, where* $\mathbf{P}_\gamma$ *determines a distribution on* Ω *with support* $\{r, b, y\} \times \{\gamma\}$*. Different models thus have disjoint supports; in fact,* Ω *is partitioned by the supports of the* $\mathbf{P}_\gamma$*,* $\gamma \in \Gamma$*.*

The general, crucial conceptual feature of identifiable models is that they are embedded in the state space, in the sense that when the state is revealed, we know the model that generated the state. Hence, the same state can not be generated by different models, shown in the example by the fact that $\mathbf{P}_\gamma$ and $\mathbf{P}_{\gamma'}$ have disjoint supports. We assume that $\mathcal{P}$ is (point)-*identifiable*, i.e., there exists a *kernel* function $k : \Omega \to \mathcal{P}$ that maps a state to the model that generated it, such that for all $\mathbf{P} \in \mathcal{P}$, $\mathbf{P}(\{\omega \in \Omega : k(\omega) = \mathbf{P}\}) = 1$. Crucial for our analysis, this means that Ω can be partitioned, with each cell of the partition, $\Omega_{\mathbf{P}} \equiv \{\omega | k(\omega) = \mathbf{P}\}$, associated with a particular model.[3] Given that Ω is finite, identifiability implies that $\mathcal{P}$ has to be finite as well. We further assume throughout that for all $\mathbf{P} \in \mathcal{P}$ and all $\omega \in \Omega_{\mathbf{P}}$, $\mathbf{P}(\omega) > 0$, i.e., $\text{supp}\,(\mathbf{P}) = \Omega_{\mathbf{P}}$.

Example 1 has a very specific feature: the state space is the product of two components, one of which completely determines the probabilistic model in play. However, this structure is not universal. The following example of an economy which will illustrate our constructs and provide motivation for the asset pricing analysis in Section 3, contains a more general case.

Running Example. *Consider an economy that may be in one of two regimes, Boom (B) or bust (b), in a given period. Estimations based on historical data associates a regime with a particular probability distribution on endowment. Over the course of a period, the endowment realizes, together with a variety of observations on the credit market, labor market, etc., which enable the NBER expert committee to determine and publicly announce the regime in operation in the period and thus, effectively, the probability distribution.*

To represent this, let Ω have three components which can be observed to take each a low or a high value: financial state ($\underline{m}, \bar{m}$), labour market ($\underline{\ell}$, $\bar{\ell}$), current endowment ($\underline{x}$,$\bar{x}$): $\Omega = \{\underline{m}, \bar{m}\} \times \{\underline{\ell}, \bar{\ell}\} \times \{\underline{x}, \bar{x}\}$. $\mathcal{P}$ has two elements $\mathbf{P}_B$ and $\mathbf{P}_b$. Say that the NBER calls a bust (resp. a Boom) when at least two variables are low (resp. high). Accordingly, $k(\underline{m}, \bar{\ell}, \underline{x}) = \mathbf{P}_b$, etc. So,

[3] In Section 2.3, we show our analysis substantially extends to the case where $\mathcal{P}$ is only *partially* or *set-identifiable*: the kernel function is set-valued and associates to each ω a subset of $\mathcal{P}$. In other words, a model is identified only up to a set of probability distributions. In this case too, the kernel function induces a partition of the state space; a cell in the partition lists states associated to a particular subset of $\mathcal{P}$ and different cells associate with disjoint subsets.

$\Omega_{\mathbf{P}_b} = \{(\underline{m}, \underline{\ell}, \underline{x}), (\underline{m}, \underline{\ell}, \bar{x}), (\underline{m}, \bar{\ell}, \underline{x}), (\bar{m}, \underline{\ell}, \underline{x})\}$, *and* $\Omega_{\mathbf{P}_B} = \Omega \setminus \Omega_{\mathbf{P}_b}$. *Notice, all three components of* Ω *are needed to identify elements in* $\mathcal{P}$.[4]

1.2 A pure exchange economy

Our pure exchange economy consists of I consumers indexed by i, consuming one good in each state of the world. Given identifiability, the standard assumption that consumption can be made contingent on states allows consumption to be made contingent on models, a crucial feature for our analysis of efficient allocations. Endowment is given by a mapping $\bar{X} : \Omega \to \mathbb{R}_+$. The fact that $(\Omega_{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}$ constitutes a partition of Ω allows us to write $\bar{X} = (\bar{X}^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}$ where $\bar{X}^{\mathbf{P}}$ is the restriction of $\bar{X}$ to $\Omega_{\mathbf{P}}$. A contingent consumption plan for consumer i is similarly defined as a mapping $X_i : \Omega \to \mathbb{R}_+$, with $X_i^{\mathbf{P}}$ its restriction to $\Omega_{\mathbf{P}}$ and $X_i = (X_i^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}$.

Running Example cont'd. *Recall, the bust (resp. Boom) regime consists of four states that are the four possible combinations of the factors, with two of them being low (resp. high). Assume both* $\mathbf{P}_b$ *and* $\mathbf{P}_B$ *are uniform, assigning probability 1/4 to each state in* $\Omega_{\mathbf{P}_b}$ *and in* $\Omega_{\mathbf{P}_B}$*, respectively. The set of possible realizations of endowments is independent of the regime and equal to* $\{\underline{x}, \bar{x}\}$ *and yet, the endowment is ambiguous because the high growth value* $\bar{x}$ *has probability 3/4 in the Boom model and probability 1/4 in a bust.*

Building on the example, say that the *range* of a mapping $f : \Omega \to \mathbb{R}$ is *model-independent* if $f(\Omega_{\mathbf{P}}) = f(\Omega_{\mathbf{Q}})$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$. Say that f is

[4]Note, we have assumed that $\mathbf{P}_B$ and $\mathbf{P}_b$ are given exogenously. They could be endogenized by using the history of the economy to estimate the distributions: assume that the probability measure over the product state space $\Omega^{\infty}_{\mathbf{P}_b}$ is i.i.d. and, analogously, that the one over $\Omega^{\infty}_{\mathbf{P}_B}$ is also i.i.d., with $\mathbf{P}_b$ (and $\mathbf{P}_B$) being the marginal on the single coordinate $\Omega_{\mathbf{P}_b}$ (respectively, $\Omega_{\mathbf{P}_B}$). Think of a single coordinate as a snapshot at a point in time. Then the empirical frequency conditional on b, and that conditional on B, identify $\mathbf{P}_b$ and $\mathbf{P}_B$, respectively. Suppose further that our example is set at a point in history when a long enough sample has been observed so that the estimates are seen to be stable and, consequently, accepted as firm. ((Klibanoff, Mukerji & Seo 2014), (Klibanoff et al. 2022), and Example 1 in (Denti & Pomatto 2022) relate such environments to ambiguity.)

unambiguous if $\mathbf{P} \circ f^{-1} = \mathbf{Q} \circ f^{-1}$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$. So, the endowment $\bar{X}$ is unambiguous if its distribution is independent of $\mathbf{P}$. Note, $\bar{X}$ is unambiguous implies that its range is model-independent but the converse is not true, as the example showed.[5]

Consumers are risk and ambiguity-averse with a common prior about model uncertainty. Consumer i's preferences are represented by the *identifiable smooth ambiguity* utility function:[6]

$$U_i(X_i) = \int_{\mathcal{P}} \phi_i\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right) \mu(d\mathbf{P}) \tag{1}$$

where $u_i : \mathbb{X}_i \to \mathbb{R}$ is assumed to be twice continuously differentiable, strictly increasing and strictly concave on its domain $\mathbb{X}_i$, an interval in $\mathbb{R}$; $\phi_i : u_i(\mathbb{X}_i) \to \mathbb{R}$ satisfies the same assumptions (with the exception that we will also consider linear ϕ_i); μ is a full support (common) prior over $\mathcal{P}$. Note, we may write $E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)$ instead of $E^{\mathbf{P}} u_i(X_i)$ as $E^{\mathbf{P}}$, the expectation operator under the probability measure $\mathbf{P}$, puts zero probability on states outside of $\Omega_{\mathbf{P}}$, due to identifiability. The Bernoulli function u_i captures risk aversion, while ϕ_i captures attitudes toward ambiguity. Consumer i is *(strictly) ambiguity-averse* if ϕ_i is (strictly) concave and *ambiguity-neutral* if ϕ_i is linear, in which case the consumer is of the expected utility type with respect to the reduced measure $\int_{\mathcal{P}} \mathbf{P}\mu(d\mathbf{P})$.

Remark 1. The fact that μ is common ensures that the risk-sharing analysis is not driven by differences in beliefs (i.e., speculation) but rather by

[5] In the running example, given that the NBER announces the regime publicly, one could use a reduced form, where the state space is simply $\{0, 1\} \times \{\underline{x}, \bar{x}\}$ with $k(0, x) = \mathbf{P}_b$, $k(1, x) = \mathbf{P}_B$, for $x = \underline{x}, \bar{x}$. This is a coarsening of the original state space. Actually, this is a general property, given identifiability: when the range of endowment is model-independent, $\bar{X}(\Omega) \times \mathcal{P}$ is a coarsening of Ω. We shall employ such a product state space in our asset pricing exercise in Section 3.

[6] The identifiable smooth representation was introduced and axiomatized by (Cerreia-Vioglio et al. 2013) taking $\mathcal{P}$ as a primitive. (Denti & Pomatto 2022) provides an axiomatic foundation where $\mathcal{P}$ is revealed by choice behavior.

differences in risk and ambiguity attitudes across consumers (i.e., insurance reasons). Taken together with heterogeneity in ϕ_i, it is similar to the condition that maxmin expected utility maximizers have *distinct* sets of priors with a non-empty intersection,[7] as the following probability matching exercise shows. Consider a bet on model $\mathbf{P}$, paying 1 on event $\Omega_{\mathbf{P}}$ and 0 off it, and a lottery ℓ^{π} which pays 1 with probability π and 0 with probability $1-\pi$. For ϕ strictly concave, there is an interval $[\underline{\pi}, \bar{\pi}] \ni \mu(\mathbf{P})$ such that for every $\pi \in [\underline{\pi}, \bar{\pi}]$, the bet on $\Omega_{\mathbf{P}}$ is less desirable than ℓ^{π} and the bet on the complementary event $\Omega \setminus \Omega_{\mathbf{P}}$ is *also* less desirable than $\ell^{1-\pi}$. Furthermore, the interval is wider, the more ambiguity-averse the consumer.[8] Hence, the consumers act *as if* their beliefs that the model $\mathbf{P}$ is true were described by a probability interval that depends on the consumer's ambiguity aversion and contains $\mu(\mathbf{P})$. Put differently, there is shared information about the likelihood of models but this information is acted upon with differing degrees of trust by heterogeneously ambiguity-averse consumers.

Running Example cont'd. *The economy here, represented with the state space $\bar{X}(\Omega) \times \mathcal{P}$ as in footnote 5, can be seen as a snapshot, at a given time, of the dynamic workhorse model in macro-finance presented in, for example, (Cecchetti, Lam & Mark 1990) where at each period there are two possible distributions, $\mathbf{P}_b$ and $\mathbf{P}_B$, corresponding to the regimes b and B. Consumers observe the regime and the Markovian transition probabilities are the consumers' conditional beliefs about the regime in the next period, which we may take to coincide with $(\mu, 1-\mu)$, where $\mu = \mu(\mathbf{P}_B)$. So, $U_i(X_i) = \mu\phi_i\left(E^{\mathbf{P}_B}u_i\left(X_i^{\mathbf{P}_B}\right)\right) + (1-\mu)\phi_i\left(E^{\mathbf{P}_b}u_i\left(X_i^{\mathbf{P}_b}\right)\right)$: consumption here is contingent on endowment and, due to identifiability, on the regime. In the traditional literature, the possibility of two regimes is treated in an*

[7] In (Billot et al. 2000) and (Rigotti, Shannon & Strzalecki 2008) it is the condition that ensures that efficiency entails full insurance under no aggregate risk.

[8] see Online-Appendix A for calculations.

ambiguity-neutral fashion, i.e., expected utility with respect to the reduced measure, $\mu\mathbf{P}_B + (1-\mu)\mathbf{P}_b$*, whereas we allow for ambiguity aversion. Note, due to the Markovian property of the endowment process, uncertainty about the forthcoming regime, and hence ambiguity to our ambiguity averse consumer, is renewed every period.*

1.3 Efficient allocations

In this section, we study properties of efficient allocations, without further restrictions on risk and ambiguity attitudes. We establish the rule, that we call "generalized Borch rule", that characterizes efficient allocations. It turns out to be necessary, but not sufficient, that efficient allocations satisfy efficiency "model-by-model".

An allocation $(X_i)_{i=1,\dots,I}$ is *feasible* if $X_i(\omega) \in \mathbb{X}_i$ for all i and $\omega \in \Omega$ with $\sum_{i=1}^{I} X_i \leq \bar{X}$. As usual, a feasible allocation $(X_i)_{i=1,\dots,I}$ is *efficient* if there is no feasible allocation $(Y_i)_{i=1,\dots,I}$ with $U_i(Y_i) \geq U_i(X_i)$ for all i, with at least one strict inequality. Let $PO(\bar{X})$ be the set of efficient allocations. Call an allocation defined on $\Omega_{\mathbf{P}}$ $\mathbf{P}$*-conditionally efficient* if it is an efficient allocation of a vNM-economy with state space $\Omega_{\mathbf{P}}$. We say that a feasible allocation (on Ω) is *conditionally efficient* if it ensures that on each $\Omega_{\mathbf{P}}$, the allocation is $\mathbf{P}$-conditionally efficient. Thus, a conditionally efficient allocation satisfies efficiency model-by-model. More formally:

Definition 1. *Say that* $(X_i^{\mathbf{P}})_{i=1,\dots,I}$ *is* $\mathbf{P}$*-feasible if* $X_i^{\mathbf{P}}(\omega) \in \mathbb{X}_i$ *for all* i*, all* $\omega \in \Omega_{\mathbf{P}}$ *and* $\sum_{i=1}^{I} X_i^{\mathbf{P}} \leq \bar{X}^{\mathbf{P}}$*. A* $\mathbf{P}$*-feasible allocation* $\left(X_i^{\mathbf{P}}\right)_{i=1,\dots,I}$ *is* $\mathbf{P}$-conditionally efficient *for* $\mathbf{P} \in \mathcal{P}$*, if there is no* $\mathbf{P}$*-feasible allocation* $\left(Y_i^{\mathbf{P}}\right)_{i=1,\dots,I}$ *with* $E^{\mathbf{P}}\left(u_i\left(Y_i^{\mathbf{P}}\right)\right) \geq E^{\mathbf{P}}\left(u_i\left(X_i^{\mathbf{P}}\right)\right)$ *for all* i*, with at least one strict inequality. A feasible allocation* $(X_i)_{i=1,\dots,I}$ *is* conditionally efficient *if* $\left(X_i^{\mathbf{P}}\right)_{i=1,\dots,I}$ *is* $\mathbf{P}$*-conditionally efficient for all* $\mathbf{P}$*.*

Letting $PO_{\Omega_{\mathbf{P}}}\left(\bar{X}^{\mathbf{P}}\right)$ be the set of $\mathbf{P}$-conditionally efficient allocations, the set of conditionally-efficient allocations is $\Pi_{\mathbf{P}\in\mathcal{P}} PO_{\Omega_{\mathbf{P}}}\left(\bar{X}^{\mathbf{P}}\right)$. To study the relationship between efficient and conditionally-efficient allocations we express them as solutions to Negishi programs. For concave utility functions, efficient allocations are the solutions to the (Negishi) problem of maximizing the weighted sum of utilities $\sum_i \lambda_i U_i\left(X_i\right)$ over all feasible allocations $\left(X_i\right)_{i=1,\ldots,I}$ for individual weights $\lambda_i \geq 0$. The resulting value function V defines the *representative consumer*'s preferences.[9]

Identifiability significantly simplifies the Negishi problem as it makes the utility functions $U_i(X_i)$ and feasibility constraint $\sum_i X_i \leq \bar{X}$ separable across models. This and the fact that the prior μ is common implies that the Negishi problem reduces to:

$$V\left(\bar{X}\right) \equiv \max_{(X_i)}\left\{\sum_i \lambda_i U_i\left(X_i\right) \text{s.t.} \sum_i X_i \leq \bar{X}\right\} = \int_{\mathcal{P}} V^{\mathbf{P}}\left(\bar{X}^{\mathbf{P}}\right)\mu(d\mathbf{P}) \quad (2)$$

where

$$\begin{array}{lll} V^{\mathbf{P}}(\bar{X}^{\mathbf{P}}) \equiv & \max\limits_{\left(X_i^{\mathbf{P}}\right)_{i=1,\ldots,I}} & \sum\limits_i \lambda_i\phi_i\left(E^{\mathbf{P}}u_i(X_i^{\mathbf{P}})\right) \\ & \text{subject to} & \sum\limits_i X_i^{\mathbf{P}} \leq \bar{X}^{\mathbf{P}}. \end{array} \quad (3)$$

Note, we may find a $\mathbf{P}$-*conditionally efficient* allocation by solving the Negishi problem in a vNM-economy with common belief $\mathbf{P}$: $\max_{(X_i^{\mathbf{P}})}\sum_i \eta_i^{\mathbf{P}} E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})$ subject to $\sum_i X_i^{\mathbf{P}} \leq \bar{X}^{\mathbf{P}}$ for *some* weights $\eta_i^{\mathbf{P}}$. The solution is independent of the common belief. A conditionally efficient allocation $(\ldots, Y^{\mathbf{P}}, Y^{\mathbf{Q}}\ldots)$ is then one where $Y^{\mathbf{P}}, Y^{\mathbf{Q}}$ solve the vNM-Negishi problem with arbitrary weights $\eta_i^{\mathbf{P}}$,$\eta_i^{\mathbf{Q}}$, respectively. Since ϕ_is are strictly increasing, solutions to (3) are also solutions to a vNM-Negishi problem. Hence, an efficient allocation is conditionally efficient, which is the content of Part 1 of Proposition 1 (the

[9]This notion of a representative consumer is common in the context of asset pricing; see e.g. Chapter 1, eqn (6) of (Duffie 2001) and should not be confused with the more demanding notion of aggregation of (Gorman 1959).

converse is not true as we argue after the proposition).[10] In vNM-economies with strictly concave utilities, efficient allocations are comonotone. Applying this logic, Part 2 of Proposition 1 shows that, conditional on each model, efficient allocations are comonotone, that is, for all $\mathbf{P} \in \mathcal{P}$, the following property, which we refer to as $\mathbf{P}$-comonotonicity, is satisfied :

$$\text{for all } i,\ j,\ \forall \omega, \omega' \in \Omega_{\mathbf{P}}, \left(X_i^{\mathbf{P}}(\omega) - X_i^{\mathbf{P}}(\omega')\right)\left(X_j^{\mathbf{P}}(\omega) - X_j^{\mathbf{P}}(\omega')\right) \geq 0.$$

Part 3 focuses on the case where endowment is unambiguous and establishes that an efficient allocation in this case provides complete insurance against model uncertainty: consumption is identical in states that have the same endowment realization and does not depend on models, just as when ϕ_i's are linear. Taking Parts 2 and 3 together, *if the endowment is unambiguous* an efficient allocation is comonotone, just as in a vNM-economy.

Proposition 1.

1. *Efficient allocations are conditionally efficient:* $PO\left(\bar{X}\right) \subset \Pi_{\mathbf{P}\in\mathcal{P}} PO_{\Omega_{\mathbf{P}}}\left(\bar{X}^{\mathbf{P}}\right)$.
2. *If* $(X_i)_{i=1,\ldots,I}$ *is an interior efficient allocation, then for any fixed* $\mathbf{P} \in \mathcal{P}$*, the allocation* $(X_i^{\mathbf{P}})_{i=1,\ldots,I}$ *is* $\mathbf{P}$*-comonotone.*[11]
3. *Assume* ϕ_i *strictly concave for all* i *and the endowment is unambiguous. Then, the allocation* $Y = (Y^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}$ *is efficient if and only if* Y *is conditionally efficient and for all* i*, all* $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$*, all* $\omega_{\mathbf{P}} \in \Omega_{\mathbf{P}}$*,* $\omega_{\mathbf{Q}} \in \Omega_{\mathbf{Q}}$*, if* $\bar{X}^{\mathbf{P}}(\omega_{\mathbf{P}}) = \bar{X}^{\mathbf{Q}}(\omega_{\mathbf{Q}})$*, then* $Y_i^{\mathbf{P}}(\omega_{\mathbf{P}}) = Y_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$*.*

The first-order conditions to (2) yield insight into properties of efficient allocations. An interior allocation $(X_i)_{i=1,\ldots,I}$ is efficient if there exist weights

[10] This is analogous to a well-known property in vNM-economies where ex ante efficiency implies interim efficiency. That is, given a partition of the state space, efficiency is preserved conditional on the realization of any constituent event.

[11] The maintained assumption that $\text{supp}\,(\mathbf{P}) = \Omega_{\mathbf{P}}$ is required there.

$\lambda_i > 0$ for all i and strictly positive multipliers $\psi^{\mathbf{P}}(\omega_{\mathbf{P}})_{\omega_{\mathbf{P}}\in\Omega_{\mathbf{P}}}$ for all $\mathbf{P}$ s.th.

$$\psi^{\mathbf{P}}(\omega_{\mathbf{P}}) = \lambda_i \phi_i' \left(E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}}\right)\right) u_i' \left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right). \tag{4}$$

Hence, whereas conditional efficiency imposes no restrictions on how the weights $\eta_i^{\mathbf{P}}$, $\mathbf{P} \in \mathcal{P}$, are related to each other, efficient allocations require an overall consistency expressed in (4). For a conditionally efficient allocation $(\ldots, Y^{\mathbf{P}}, Y^{\mathbf{Q}} \ldots)$ to be efficient, the associated $\eta_i^{\mathbf{P}}$,$\eta_i^{\mathbf{Q}}$ have to satisfy $\lambda_i \phi_i' \left(E^{\mathbf{P}} u_i \left(Y_i^{\mathbf{P}}\right)\right) = \eta_i^{\mathbf{P}}$ and $\lambda_i \phi_i' \left(E^{\mathbf{Q}} u_i \left(Y_i^{\mathbf{Q}}\right)\right) = \eta_i^{\mathbf{Q}}$. Risk-sharing in an ambiguity-neutral economy (linear ϕ_is) entails $\eta_i^{\mathbf{P}} = \eta_i^{\mathbf{Q}} = \lambda_i$, implying $Y_i^{\mathbf{P}} = Y_i^{\mathbf{Q}}$. In contrast, under ambiguity aversion, model-specific weights, $\eta_i^{\mathbf{P}}$, are "adjusted" so that weights are higher on a consumer at models where her expected utility (given the model) is lower and vice versa, ensuring that allocations across models adjust. This is because consumers value smoothing expected utility across models, as captured by the terms ϕ_i' in the *generalized Borch rule* that follows from (4):

$$\frac{\phi_i'(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}}))u_i'(X_i^{\mathbf{P}}(\omega_{\mathbf{P}}))}{\phi_i'(E^{\mathbf{Q}} u_i(X_i^{\mathbf{Q}}))u_i'(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}}))} = \frac{\phi_j'(E^{\mathbf{P}} u_j(X_j^{\mathbf{P}}))u_j'(X_j^{\mathbf{P}}(\omega_{\mathbf{P}}))}{\phi_j'(E^{\mathbf{Q}} u_j(X_j^{\mathbf{Q}}))u_j'(X_j^{\mathbf{Q}}(\omega_{\mathbf{Q}}))} \tag{5}$$

Applied to two states within $\Omega_{\mathbf{P}}$, the rule boils down to the standard Borch rule. When applied to two states associated with different models, it embeds uncertainty sharing beyond standard risk sharing.

Running Example cont'd. *Specify endowment values, say* $\bar{x} = 10$ *and* $\underline{x} = 4$, *and let* $\bar{x}$ *have probability 3/4 under* $\mathbf{P}_B$ *and 1/4 under* $\mathbf{P}_b$. *There are two consumers, each with Bernoulli utility function* $u(x) = \log(x)$. *Under such assumptions, it is well-known that, conditional on a model, the efficient sharing rule specifies that each consumer gets a (constant) share of the (random) endowment. Suppose that this share were the same in each model, say,* $\theta_1^{\mathbf{P}_B} = 1/4 = \theta_1^{\mathbf{P}_b}$ *(and hence* $\theta_2^{\mathbf{P}_b} = 3/4 = \theta_2^{\mathbf{P}_B}$*). This allocation would satisfy the usual Borch rule. It would not satisfy the generalized Borch rule* (5) *unless*

$$\frac{\phi_1'(E^{\mathbf{P}_B}\log((1/4)\bar{X}))}{\phi_1'(E^{\mathbf{P}_b}\log((1/4)\bar{X}))} = \frac{\phi_2'(E^{\mathbf{P}_B}\log((3/4)\bar{X}))}{\phi_2'(E^{\mathbf{P}_b}\log((3/4)\bar{X}))},$$

a condition that generically does not hold (unless consumers are ambiguity neutral); hence, a rule that gives consumers a share of endowment which does not depend on the model will not be efficient.

Note that only a specific dependence of the share on the model will ensure efficiency. Indeed, consider some arbitrary allocation where the share does depend on the model, taken to be $\theta_1^{\mathbf{P}_B} = 1/4 = \theta_2^{\mathbf{P}_b}$ *and* $\theta_1^{\mathbf{P}_b} = 3/4 = \theta_2^{\mathbf{P}_B}$*. The corresponding allocation is conditionally efficient but is not efficient. Direct computation yields that* $E^{\mathbf{P}_B}\log(X_1^{\mathbf{P}_B}) < E^{\mathbf{P}_b}\log(X_1^{\mathbf{P}_b})$ *while the reverse inequality holds for 2, causing* (5) *to be violated, leaving unrealized efficiency gains (with strictly concave* ϕ_i*s):*

$$\frac{\phi_1'(E^{\mathbf{P}_B}u_1'(X_1^{\mathbf{P}_B}))u_1'(\frac{1}{4}\bar{x})}{\phi_1'(E^{\mathbf{P}_b}\log(X_1^{\mathbf{P}_b}))u_1'(\frac{3}{4}\bar{x})} > 1 > \frac{\phi_2'(E^{\mathbf{P}_B}u_2(X_2^{\mathbf{P}_B}))u_2'(\frac{3}{4}\bar{x})}{\phi_2'(E^{\mathbf{P}_b}u_2(X_2^{\mathbf{P}_b}))u_2'(\frac{1}{4}\bar{x})}$$

(a similar inequality holds for $\underline{x}$*.)*

Unlike in standard expected utility models, the Borch rule does not apply when ϕ_i's are strictly concave. As a result, efficient allocations are not necessarily comonotone. However, ambiguity-averse consumers seek to smooth expected utility across models, leading to a related property we term *Expected-Utility Comonotonicity.* Intuitively, if at an allocation a consumer i's expected utility on event $\Omega_{\mathbf{P}}$ were higher than on event $\Omega_{\mathbf{Q}}$ while it is the reverse for j, transfers may be possible that bring their expected utilities contingent on those two events closer. If such transfers can be effected then we may expect efficient allocations to satisfy the property that the ordering of i's expected utility across $\Omega_{\mathbf{P}}$ and $\Omega_{\mathbf{Q}}$ is the same as j's.

Definition 2. *An allocation* $(X_i)_{i=1,\ldots,I}$ *is* Expected-Utility-comonotone *(or EU-comonotone) if for every* i, j *and* $\mathbf{P},\mathbf{Q} \in \mathcal{P}$, $E^{\mathbf{P}}u_i(X_i^{\mathbf{P}}) \leq E^{\mathbf{Q}}u_i(X_i^{\mathbf{Q}})$ *if and only if* $E^{\mathbf{P}}u_j(X_j^{\mathbf{P}}) \leq E^{\mathbf{Q}}u_j(X_j^{\mathbf{Q}})$.

The following proposition establishes that efficient allocations are EU-comonotone whenever the distributions on (aggregate) endowment induced by models are ordered by first-order stochastic dominance (FOSD).

Proposition 2. *Assume the range of $\bar{X}$ is model-independent and ϕ_i strictly concave for all i. Let $(X_i)_{i=1,\dots,I}$ be an efficient allocation. If the set $\{\mathbf{P}\circ\bar{X}^{-1} \mid \mathbf{P} \in \mathcal{P}\}$ is totally ordered by FOSD, then $(X_i)_{i=1,\dots,I}$ is EU-comonotone.*

Furthermore, under some technical conditions, if an allocation is conditionally efficient and satisfies EU-comonotonicity given a profile $(u_i)_{i=1,\dots,I}$, then there exists a profile of concave and twice-differentiable $(\phi_i)_{i=1,\dots,I}$ such that the allocation is efficient.[12]

2 Representative consumer and sharing rules

We now identify economies that admit a *smooth ambiguity representative consumer*. In such economies, we characterize efficient sharing rules and show how risk and ambiguity attitudes of the representative consumer relate to those of the individual consumers. In Section 2.3, we show that the insights obtained are robust in the sense that they extend to the case where models are only *set-identified*.

To characterize the representative consumer, we turn to an equivalent writing of the identifiable smooth ambiguity representation. Let $c_i^{\mathbf{P}}(X_i^{\mathbf{P}}) = u_i^{-1}\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right)$ be the certainty equivalent of consumer i's consumption plan under model $\mathbf{P}$, and let $v_i : \mathbb{X}_i \to \mathbb{R}$ be defined by $v_i = \phi_i \circ u_i$. The smooth ambiguity utility (1) can be written equivalently as an expected utility of certainty equivalents:

$$U_i\left(X_i\right) = \int_{\mathcal{P}} v_i\left(c_i^{\mathbf{P}}(X_i^{\mathbf{P}})\right) \mu(d\mathbf{P}). \tag{6}$$

[12] Proposition 10 in (Hara et al. 2022).

Combining (6) with program (2) we can rewrite the representative consumer's preferences as:

$$V\left(\bar{X}\right)=\int_{\mathcal{P}}\max_{c\in\mathcal{C}^{\mathbf{P}}}\sum_{i}\lambda_i v_i(c_i)\mu(d\mathbf{P}) \tag{7}$$

where $\mathcal{C}^{\mathbf{P}}=\left\{c\in\mathbb{R}^I : c_i\le c_i^{\mathbf{P}}(X_i^{\mathbf{P}}) \text{ for some } X^{\mathbf{P}}\in PO_{\Omega_{\mathbf{P}}}(\bar{X}^{\mathbf{P}})\right\}$.

The set $\mathcal{C}^{\mathbf{P}}$ collects certainty equivalents corresponding to **P**-conditionally efficient allocations. The subset of $\mathcal{C}^{\mathbf{P}}$ that corresponds to efficient allocations consists of the ones that, furthermore, maximize $\sum_i \lambda_i v_i(c_i)$. Given that the problem (7) is separable across models, this maximization can be done model-by-model. Hence, there is an algorithm for the social planner to identify $V(\bar{X})$ and efficient allocations. However, without further assumptions, $V\left(\bar{X}\right)$ is not necessarily of the smooth ambiguity type.

2.1 A smooth ambiguity representative consumer

Recall that both the Bernoulli utility function u_i as well as the ambiguity index v_i are concave and increasing functions. A twice differentiable concave and increasing function f is said to exhibit hyperbolic absolute risk aversion (HARA), or linear risk tolerance, if it satisfies

$$-\frac{f'(x)}{f''(x)}=a+bx \tag{8}$$

for the parameter pair (a,b), $a\in\mathbb{R}$ and $b>0$. The HARA class covers the gamut of Bernoulli utility functions considered in economics and finance.[13] Under risk, it is well-known that efficient risk sharing rules are linear if and only if consumers' utility functions exhibit linear risk tolerance with common marginal risk tolerance (henceforth CMRT).[14]

[13]For $b=0$, the function is CARA with index $\frac{1}{a}$. Quadratic functions correspond to $b=-1$. When $b>0$ and $a=0$ the function exhibits CRRA. When $b>0$ and $a\neq 0$, the class of functions is of the "shifted power" type, (Back 2017) Section 1.3.

[14]See Proposition 16.13 in (Magill & Quinzii 1996), based on (Wilson 1968) and (Cass & Stiglitz 1970) and more recently (Hara, Huang & Kuzmics 2007).

Condition 1. *The Bernoulli utility functions* $(u_i)_{i=1,\ldots,I}$ *are HARA with parameters* $(a_i, b_i)_{i=1,\ldots,I}$ *and satisfy CMRT, that is,* $b_i = b$ *for all* i*.*

In what follows, we revisit problem (7) and show that there exists a smooth ambiguity representative consumer if and only if the consumers' utilities satisfy Condition 1. To find (certainty equivalents corresponding to) **P**-conditionally efficient allocations, one has to maximize a weighted sum of Bernoulli utilities for weights $\lambda_i \geq 0$:

$$\begin{array}{rl} u(\bar{x}) \equiv \max\limits_{(x_i)_{i=1,\ldots,I}} & \sum\limits_{i=1,\ldots,I} \lambda_i u_i(x_i) \\ \text{subject to} & \sum\limits_{i=1,\ldots,I} x_i \leq \bar{x}. \end{array} \tag{9}$$

Under Condition 1, the value function u does not depend on the weights λ_i.[15] Let $c_u^{\mathbf{P}}(\bar{X^{\mathbf{P}}}) = u^{-1}\left(E^{\mathbf{P}} u(\bar{X}^{\mathbf{P}})\right)$ denote the certainty equivalent of the representative consumer under model P ("aggregate certainty equivalent at **P**" for short).[16] It turns out, under Condition 1, **P**-conditional efficient allocations are precisely those feasible allocations for which the individual certainty equivalents sum up to the aggregate certainty equivalent under **P**.

Lemma 1. *Suppose Condition 1 holds. Then,* $(X_i^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P},i=1,\ldots,I}$ *is conditionally efficient if and only if* $(X_i^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P},i=1,\ldots,I}$ *is feasible and* $\sum_{i=1,\ldots,I} c_i^{\mathbf{P}}(X_i^{\mathbf{P}}) = c_u^{\mathbf{P}}(\bar{X})$ *for all* $\mathbf{P}$*.*

Lemma 1 allows us to replace $\mathcal{C}^{\mathbf{P}}$ in (7) by a linear constraint. This makes the social planner's problem (7) analogous to one of implementing efficiency in a vNM-economy. Now, model-by-model, we can think of the planner's problem as one of efficiently allocating the aggregate resource $c_u^{\mathbf{P}}(\bar{X}^{\mathbf{P}})$ in

[15] (LeRoy & Werner 2014), Section 16.8. or (Gollier 2001), section 21.4.1.

[16] The characterization of the representative consumer's utility u is in (Wilson 1968) and (Cass & Stiglitz 1970). It is in the same HARA subclass as the individuals', e.g., when u_is are shifted power with parameters (a_i, b), u is shifted power with parameters $(\sum_i a_i, b)$.

each "state $\mathbf{P}$" across consumers with "Bernoulli utility v_i", as the following Negishi program expresses:

$$\begin{aligned} v(c_u^{\mathbf{P}}(\bar{X}^{\mathbf{P}})) \equiv \max_{(c_i)_{i=1,\dots,I}} & \sum_{i=1,\dots,I} \lambda_i v_i\left(c_i\right) \\ \text{subject to} & \sum_{i=1,\dots,I} c_i \leq c_u^{\mathbf{P}}(\bar{X}^{\mathbf{P}}). \end{aligned} \tag{10}$$

Analogous to the solution of a vNM-economy problem, the solution $\left(c_i^{\mathbf{P}}(X_i^{\mathbf{P}})\right)_{i=1,\dots,I}$ is comonotone with respect to the aggregate certainty equivalent. Hence, given that u_i^{-1} is strictly increasing for all i, the efficient allocation $(X_i)_{i=1,\dots,I}$ is EU-comonotone. Finally, notice that $V^{\mathbf{P}}(\bar{X}^{\mathbf{P}}) = v(c_u^{\mathbf{P}}(\bar{X}^{\mathbf{P}})) = v(u^{-1}\left(E^{\mathbf{P}}u(\bar{X}^{\mathbf{P}})\right))$. Hence, letting $\phi = v \circ u^{-1}$, we can rewrite the representative consumer's utility as $V\left(\bar{X}\right) = \int_{\mathcal{P}} \phi\left(E^{\mathbf{P}}u\left(\bar{X}^{\mathbf{P}}\right)\right)\mu(d\mathbf{P})$.

Proposition 3. *Let $(u_i)_{i=1,\dots,I}$ satisfy Condition 1. Then,*

1. *The representative consumer's utility V is of the smooth ambiguity form: $V\left(\bar{X}\right) = \int_{\mathcal{P}} \phi\left(E^{\mathbf{P}}u\left(\bar{X}^{\mathbf{P}}\right)\right)\mu(d\mathbf{P})$ with $\phi = v \circ u^{-1}$, where u is the value function of* (9) *and v is the value function of* (10)*. Moreover, $\phi'' \leq 0$, and $\phi'' = 0$ if and only if $\phi_i'' = 0$ for all i.*

2. *Assume ϕ_is are strictly concave, then an efficient allocation is EU-comonotone.*

A consequence of Part 1 of Proposition 3 is that the representative consumer is strictly ambiguity-averse as soon as there is one strictly ambiguity-averse consumer in the economy: an interesting departure from the case of pure risk-sharing, where if some consumers were risk-neutral they would bear all the risk and the representative consumer would be risk-neutral. By Part 2, all consumers rank models in the same way at an efficient allocation; unlike in Proposition 2, this result obtains *without restrictions on $\mathcal{P}$*.

Condition 1 is not only sufficient for obtaining a smooth ambiguity representative consumer, but also necessary: if it were not satisfied we can find an economy with heterogeneous smooth ambiguity-averse consumers (with common μ) such that the representative consumer's utility function is not of the smooth ambiguity type.

Proposition 4. *Assume* $|\Omega| \geq 4$ *and suppose the profile* $(u_i)_{i=1,\ldots,I}$ *does not satisfy Condition 1. Then, there are* μ, $(\phi_i)_{i=1,\ldots,I}$ *and* $\bar{X}$ *such that, if* V *is defined as in* (2)*, there is no pair* (u, ϕ) *such that* $V\left(\bar{Y}\right) = \int_{\mathcal{P}} \phi\left(E^{\mathbf{P}} u\left(\bar{Y}^{\mathbf{P}}\right)\right) \mu(d\mathbf{P})$ *for all* $\bar{Y}$.

An intuition for Proposition 4 is as follows. Recall, the restriction of any efficient allocation Y to $\Omega_{\mathbf{P}}$ is a solution of a vNM-Negishi problem with model-**P**-specific consumer weights $\eta_i^{\mathbf{P}}$, where $\eta_i^{\mathbf{P}} = \lambda_i \phi_i'\left(E^{\mathbf{P}} u_i\left(Y_i^{\mathbf{P}}\right)\right)$. In general, unless Condition 1 holds, the representative consumer's Bernoulli utility, obtained from the model-**P**-specific vNM-Negishi program, is dependent on consumer weights $\eta_i^{\mathbf{P}}$. Hence, unless Condition 1 holds, the representative consumer's Bernoulli utility will be model-specific, in general. This is incompatible with a smooth ambiguity representation (1) where the representative consumer's Bernoulli utility has to be the same across models.

2.2 Sharing rules and aggregate ambiguity aversion

Next, we focus on finding sharing rules in economies that admit a smooth ambiguity representative consumer, i.e., economies satisfying Condition 1. Since v_i and u_i are both defined on the consumption space, it is natural to require v_i to have the same parametric form as u_i. Imposing that $(v_i)_{i=1,\ldots,I}$ is HARA, further to Condition 1 on $(u_i)_{i=1,\ldots,I}$, enables us to characterize $\phi = v \circ u^{-1}$, the ambiguity aversion of the representative consumer, as a function of the individual consumers' ambiguity aversion. We consider here the case where u_i's and v_i's have strictly positive marginal risk tolerance, which is the traditional focus in macro and finance models.

Condition 2. *There exist $\alpha > 0$, ζ_i and $\gamma_i \geq \alpha$ for all i s.th. u_i and v_i are defined on the shifted interval $\mathbb{X}_i = (\zeta_i, \infty)$ by:*

$$u_i(x_i) = \begin{cases} \frac{\alpha}{1-\alpha}\left(\frac{x_i-\zeta_i}{\alpha}\right)^{1-\alpha} & \text{if } \alpha \neq 1 \\ \ln(x_i - \zeta_i) & \text{if } \alpha = 1 \end{cases} \quad v_i(x_i) = \begin{cases} \frac{\gamma_i}{1-\gamma_i}\left(\frac{x_i-\zeta_i}{\gamma_i}\right)^{1-\gamma_i} & \text{if } \gamma_i \neq 1 \\ \ln(x_i - \zeta_i) & \text{if } \gamma_i = 1 \end{cases}$$

Under Condition 2, $-\frac{u_i''(x)}{u_i'(x)} = \frac{\alpha}{x-\zeta_i}$. Hence, the relative risk aversion coefficient, relative to effective consumption $z \equiv x - \zeta_i$, is α. Define i's *relative ambiguity aversion* coefficient, relative to effective consumption by[17]

$$RAA_{\phi_i}(z) \equiv -\frac{\phi_i''(u_i(z+\zeta_i))}{\phi_i'(u_i(z+\zeta_i))} u_i'(z+\zeta_i) z.$$

Under Condition 2, $RAA_{\phi_i}(z) = \gamma_i - \alpha$ is positive for all i and independent of z. Notice that Condition 2 does *not* require the v_i's to satisfy common marginal risk tolerance. This is significant as it allows for heterogeneity in the consumers' relative ambiguity aversion.

Proposition 5 first characterizes the efficient sharing rule.[18] *Conditional on a model*, it is linear. However, the slope coefficient of the linear sharing rule is model-dependent.

Running Example cont'd. *Assume $v_i = \frac{x_i^{1-\gamma_i}}{1-\gamma_i}$ with $\gamma_1 > \gamma_2 > 1$ and $u_i = \log$. The generalized Borch rule determines the model-contingent shares $\theta_i^{\mathbf{P}}$ of aggregate endowment that i gets:*

$$\left(\frac{\theta_1^{\mathbf{P}_b}}{\theta_1^{\mathbf{P}_B}}\right)^{\gamma_1} \frac{e^{(1-\gamma_1)E^{\mathbf{P}_B} log(\bar{X})}}{e^{(1-\gamma_1)E^{\mathbf{P}_b} log(\bar{X})}} = \left(\frac{\theta_2^{\mathbf{P}_b}}{\theta_2^{\mathbf{P}_B}}\right)^{\gamma_2} \frac{e^{(1-\gamma_2)E^{\mathbf{P}_B} log(\bar{X})}}{e^{(1-\gamma_2)E^{\mathbf{P}_b} log(\bar{X})}}$$

Hence, since $E^{\mathbf{P}_B} log(\bar{X}) > E^{\mathbf{P}_b} log(\bar{X})$ and $\gamma_1 > \gamma_2 > 1$,

$$\left(\frac{\theta_1^{\mathbf{P}_b}}{\theta_1^{\mathbf{P}_B}}\right)^{\gamma_1} > \left(\frac{\theta_2^{\mathbf{P}_b}}{\theta_2^{\mathbf{P}_B}}\right)^{\gamma_2}.$$

[17] See Online-Appendix B.

[18] Online-Appendix E contains further material describing the efficient rule.

This implies that $\theta_1^{\mathbf{P}_b} > \theta_1^{\mathbf{P}_B}$ *and* $\theta_2^{\mathbf{P}_b} < \theta_2^{\mathbf{P}_B}$ *(since* $\theta_1^{\mathbf{P}_b} + \theta_2^{\mathbf{P}_b} = 1$ *and* $\theta_1^{\mathbf{P}_B} + \theta_2^{\mathbf{P}_B} = 1$*). A consumer's optimal share thus varies with the model and is dictated by her relative ambiguity aversion* $\gamma_i - 1$*. The less ambiguity averse consumer (2) holds a larger share of the endowment in a Boom than in a bust, while the more ambiguity averse consumer (1) holds a larger share in a bust than in a Boom. Said differently, the ratio of the share of the more ambiguity-averse to the share of the less ambiguity-averse consumers is higher during a bust than during a Boom:* $\frac{\theta_1^{\mathbf{P}_b}}{\theta_2^{\mathbf{P}_b}} > \frac{\theta_1^{\mathbf{P}_B}}{\theta_2^{\mathbf{P}_B}}$ *(this ratio would be constant if ambiguity aversion were homogeneous). Thus, relatively more of the resources will be allocated to relatively more ambiguity-averse consumers in a bust than in a Boom, making the representative consumer more ambiguity-averse in a bust. A further point to note is that efficient allocations are not comonotone, since* $\theta_1^{\mathbf{P}_B} < \theta_1^{\mathbf{P}_b}$ *while* $\theta_2^{\mathbf{P}_B} > \theta_2^{\mathbf{P}_b}$*. Consider two states, one in* $\Omega_{\mathbf{P}_b}$ *and one in* $\Omega_{\mathbf{P}_B}$*, both with the same endowment realization; then, 1 (resp. 2) has higher consumption in the former (resp. latter) state, violating comonotonicity.*

Part 1 of Proposition 5 establishes that the slope coefficient of the sharing rule is a function of the aggregate certainty equivalent and describes key properties of that function. Part 2 characterizes the smooth ambiguity representative consumer's relative ambiguity aversion, denoted RAA_ϕ.[19]

Proposition 5. *Assume the range of* $\bar{X}$ *is model-independent. Let* $(X_i)_{i=1,\ldots,I}$ *be an efficient allocation. Then, under Condition 2:*

1. *there exist functions* $\theta_i : (0, \infty) \to (0, 1)$ *with* $\sum_i \theta_i(z) = 1$*, such that*

$$X_i^{\mathbf{P}} = \theta_i(c_u^{\mathbf{P}}(\bar{X}) - \zeta) \cdot (\bar{X} - \zeta) + \zeta_i$$

where $\zeta = \sum_i \zeta_i$*. Furthermore,* $\forall i, j$*, and* $\forall z > 0$*,*

[19]The value function v corresponding to the Negishi program (10) is not independent of the weights (λ_i)'s since v_i's may not have CMRT. However, the property we establish about the representative consumer's relative ambiguity aversion holds irrespective of the specification of these weights.

(a) $\frac{d}{dz}\left(\frac{\theta_j(z)}{\theta_i(z)}\right) > (=)\ 0$ *iff* $RAA_{\phi_i} = \gamma_i - \alpha > (=)\ RAA_{\phi_j} = \gamma_j - \alpha$;

(b) $\theta_i'(z) > (=)\ 0$ *iff* $RAA_\phi(z) > (=)\ RAA_{\phi_i}$.

2. $RAA_\phi(z) = \left[\sum_i \theta_i(z) \frac{1}{\gamma_i}\right]^{-1} - \alpha$, *and it is strictly decreasing with* z *if* $\min_i \gamma_i < \max_i \gamma_i$ *that is, if relative ambiguity aversion is heterogeneous in the economy. It is constant if* $\min_i \gamma_i = \max_i \gamma_i$.

Let i be more relatively ambiguity-averse than j. Then, Part 1 (a) shows that i's share relative to j's decreases as we go to better models, i.e., with higher aggregate certainty equivalents. Hence, the more relatively ambiguity-averse consumer has a smoother expected utility across models. Part 1 (b) shows that as we move from worse to better models, a consumer more (resp. less) relatively ambiguity-averse than the representative consumer will see her share decrease (resp. increase). Finally, if relative ambiguity aversion were homogeneous, then θ_i would be a constant function. Hence, efficient allocations under ambiguity are different from those under expected utility *only* when ambiguity attitudes are heterogeneous *and* the endowment is ambiguous (if not, $c_u^{\mathbf{P}}(\bar{X})$ would be constant across models).

Part 2 shows that the non-constant term in $RAA_\phi(z)$ is a weighted harmonic mean of the γ_i's, weighted by i's share of the aggregate certainty equivalent at an efficient allocation. Together with Part 1 (a), this implies that as we go to better models, the representative consumer's relative ambiguity aversion is influenced more by consumers with lower relative ambiguity aversion. Thus, if (and only if) consumers have heterogeneous relative ambiguity aversion, the relative ambiguity aversion of the representative consumer declines as models get better. Remarkably, even though individual consumers have constant relative ambiguity aversion, at any efficient allocation the representative consumer has decreasing relative ambiguity aversion.[20]

[20] We extend the analysis to the cases of CARA u_i's and v_i's in Online-Appendix D

2.3 Robustness to set-identifiability

(Denti & Pomatto 2020) provides an axiomatization for *partially-identifiable* preferences, where $\mathcal{P}$ is not point-identified but only set-identified, that is, the kernel associates to each ω a *set* of probability laws. The functional representing such preferences is

$$U_i(X_i) = \int_{\mathcal{M}} \phi_i \left(\min_{\mathbf{P}\in\mathbf{M}} E^{\mathbf{P}} u_i \left(X_i^{\mathbf{M}} \right) \right) \mu(d\mathbf{M}), \tag{11}$$

where $\mathcal{M}$ is the set of set-identified models, i.e., a collection of sets $\mathbf{M}$, each $\mathbf{M}$ being a set of probability laws. Within each $\mathbf{M} \in \mathcal{M}$, the decision maker is maxmin expected utility (MEU) and then aggregates, over $\mathcal{M}$, the MEU-utilities via the smooth ambiguity aggregator ϕ_i.

(Wakai 2007) showed that when consumers have MEU preferences satisfying Condition 1, the Pareto optimal allocations are comonotone and there is a representative consumer with HARA utility function. Moreover, within each set $\mathbf{M}$, at an efficient allocation, the (set of) minimizing probability law(s) is the same for all individuals.[21] Armed with this result, we can "replace" each set $\mathbf{M}$ by the worst probability law in $\mathbf{M}$, apply our analysis to this economy and obtain results analogous to those we have in the point-identified case. That is, Propositions 3 and 5 continue to hold, though efficient allocations are now contingent on $\mathbf{M} \in \mathcal{M}$, whereas previously they were contingent on $\mathbf{P} \in \mathcal{P}$. In this sense, the key insights of our analysis of the representative consumer and sharing rule in economies satisfying Condition 1 remain robust to partial/set-identification.

–there, the representative consumer has constant relative ambiguity aversion. When u_i's and v_i's are quadratic, ϕ_i is linear and we are in a vNM-economy.

[21] One may define $\mathbf{M}$-conditionally efficient allocations analogous to the definition of $\mathbf{P}$-conditionally efficient allocations for the point-identified case. More precisely, Wakai's result ensures that at an $\mathbf{M}$-conditionally efficient allocation, consumers' utilities are affinely related.

3 Pricing kernel

In this section, we explore the asset pricing implications of heterogeneous ambiguity aversion. Specifically, we use the features of the representative consumer described in Proposition 5 to derive the properties of the pricing kernel. The macro-finance literature that has used the smooth ambiguity model (for instance, (Ju & Miao 2012), (Collard et al. 2018), (Hansen & Miao 2018), (Hansen & Miao 2022), (Gallant, Jahan-Parvar & Liu 2019), (Thimme & Volkert 2015) and (Huo, Pedroni & Pei 2024)) assumed constant relative ambiguity aversion for the representative consumer. As we showed, this corresponds to homogeneity of ambiguity aversion in the underlying economy. Allowing for heterogeneity, and hence a representative consumer with decreasing relative ambiguity aversion, brings the shape of the pricing kernel closer to its documented empirical regularities.

As explained in footnote 5, under identifiability and model-independence of the range of endowment, there exists a coarsening of Ω that can be written as $\bar{X}(\Omega)\times\mathcal{P}$. We assume that $\bar{X}(\Omega)=\mathbb{R}_{++}$.[22] Let $P(x)=\mathbf{P}(\{\omega\in\Omega|\bar{X}(\omega)\leq x\})$ be the cumulative distribution function of endowment under model $\mathbf{P}$ and let $p(x)$ be the associated density with respect to the Lebesgue measure.

Assuming a complete set of securities, we derive asset price properties. From (4), the price density for (x,P)-contingent claims in equilibrium is given by a function $\psi:\bar{X}(\Omega)\times\mathcal{P}\to\mathbb{R}_{++}$ such that

$$\psi(x,\mathbf{P})=\phi'\left(E^{\mathbf{P}}u\left(\bar{X}^{\mathbf{P}}\right)\right)p(x)u'\left(\bar{X}^{\mathbf{P}}(x)\right). \tag{12}$$

An x-contingent claim delivers a unit of the good if x occurs, no matter what $\mathbf{P}$ is, and hence its price density is the sum over models in $\mathcal{P}$ of the price of $(x,\mathbf{P})$-contingent claims. Divide this price by the density of x with respect to

[22] Note that here, we take $\bar{X}(\Omega)$, and hence Ω, to be infinite.

the reduced measure, $p^{\mu}(x) = \int_{\mathcal{P}} q(x)\mu(d\mathbf{Q})$, to obtain the *pricing kernel*:[23]

$$x \mapsto \pi_{u,\phi}(x) \equiv \int_{\mathcal{P}} \frac{p(x)}{p^{\mu}(x)} \phi' \left(E^{\mathbf{P}} u \left(\bar{X} \right) \right) u'(x) \mu(d\mathbf{P}). \tag{13}$$

The pricing kernel under ambiguity aversion is a weighted average of marginal utilities $u'(x)$ with weights $\frac{p(x)}{p^{\mu}(x)} \phi' \left(E^{\mathbf{P}} u(\bar{X}) \right)$ whereas under ambiguity-neutrality it is simply the marginal utility. The price of any contingent claim y written on endowment is $E\left[\pi_{u,\phi} y\right] = \int_{\mathbb{R}_{++}} \pi_{u,\phi}(x) y\left(x\right) p^{\mu}(x) dx$.

The *elasticity of the pricing kernel* $\pi_{u,\phi}$ at x, given by $\varepsilon(x; \pi_{u,\phi}) \equiv -\frac{\pi'_{u,\phi}(x)x}{\pi_{u,\phi}(x)}$, is a measure of the kernel's variability. The Hansen-Jagannathan (H-J) bound of the pricing kernel $\pi_{u,\phi}$ is equal to $\sigma\left[\pi_{u,\phi}\right] / E\left[\pi_{u,\phi}\right]$ where σ denotes the standard-deviation. It is, in principle, deducible from returns data. A theory that delivers a higher H-J bound has a greater potential to accommodate market volatility and large equity premia. In what follows, we give two illustrations of how the elasticity of the pricing kernel and the H-J bound are affected by ambiguity aversion. In the first, we assume a Gaussian environment which allows us to obtain an analytical characterization. In the second, we return to the running example.

Assumption 1.

1. *For each* $\mathbf{P}$*,* $\bar{X}^{\mathbf{P}}$ *is log-normally distributed,* $log(\bar{X}^{\mathbf{P}}) \sim \mathcal{N}(m_{\mathbf{P}}, \sigma^2)$ *and* $\mathcal{P}$ *is the set of all such log-normal distributions.*

2. *The prior on the parameters* $m_{\mathbf{P}} \in \mathbb{R}$ *is* $\mathcal{N}(\hat{m}, \hat{\sigma}^2)$*.*

Assumption 1 ensures, firstly, that an ambiguity-neutral consumer believes that endowment is log-normal, a common assumption in the macro-finance literature. Secondly, it assumes model uncertainty to be uncertainty about the mean of the distribution of the growth rate.

[23] If the representative consumer were both risk and ambiguity-neutral, the pricing kernel would be constant.

Running Example cont'd. *In (Cecchetti, Lam & Mark 1990) the specification for the probability distributions of x in the two regimes are log-normals with a common variance, with bust having a lower mean than Boom. (Kandel & Stambaugh 1991) additionally allow the two regimes to have different variances. (Ju & Miao 2012), in their dynamic Lucas tree model, also use such a Markovian endowment process.*

Proposition 6 allows us to compare properties of the pricing kernel and H-J bounds across two economies: one with homogeneous relative ambiguity aversion (where RAA_ϕ is constant) and the other with heterogeneous relative ambiguity aversion (where RAA_ϕ is decreasing). Part 1 shows how the elasticity of the pricing kernel varies with endowment and how the H-J bounds react, in the homogeneous economy, to changes in $\hat{m}$, the beliefs about the mean growth rate. Part 2 does the same for the heterogeneous economy.

Proposition 6. *Suppose Assumption 1 holds, u is CRRA and $\phi''(.) < 0$.*

1. *If RAA_ϕ is constant, then $\varepsilon(x; \pi_{u,\phi})$ is constant and $\dfrac{\sigma^{\hat{m}}(\pi_{u,\phi}(., \hat{m}))}{E^{\hat{m}}(\pi_{u,\phi}(., \hat{m}))}$ is constant in $\hat{m}$.*

2. *If RAA_ϕ is strictly decreasing, then $\varepsilon(x; \pi_{u,\phi})$ is strictly decreasing in x and $\dfrac{\sigma^{\hat{m}}(\pi_{u,\phi}(., \hat{m}))}{E^{\hat{m}}(\pi_{u,\phi}(., \hat{m}))}$ is strictly decreasing in $\hat{m}$.*

In Part 1, let the ϕ correspond to a homogeneous multi-consumer economy where $u_i = u$ and $v_i = v$ with the CRRA coefficients α and γ respectively. Then, the elasticity of the pricing kernel is constant and, in fact, may be described explicitly: $\varepsilon(x, \pi_{u;\phi}) = \frac{\sigma^2}{\sigma^2+\hat{\sigma}^2}\alpha + \frac{\hat{\sigma}^2}{\sigma^2+\hat{\sigma}^2}\gamma$. Evidently, then, the higher the consumer's ambiguity aversion $\gamma - \alpha$, or the higher the ambiguity, in the sense of a larger $\hat{\sigma}^2$, the higher the elasticity of the kernel. In Part 2, the ϕ corresponds to a heterogeneously ambiguity-averse economy. Assume the relative ambiguity aversion in the homogeneous economy of Part 1 lies

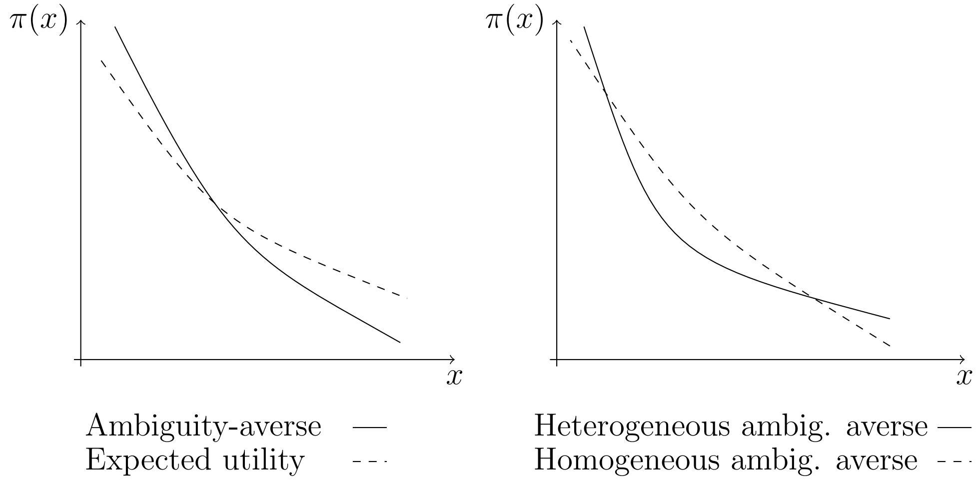

Figure 1: Pricing kernels

strictly between the maximum and minimum relative ambiguity aversion in the heterogeneously ambiguity-averse economy and normalize the pricing kernels so that the two economies have the same bond price (where a bond delivers a unit of the good in all states). Then, the kernels have exactly two points of intersection and have the qualitative features depicted in the right panel of Figure 1.[24] For low (resp. high) realizations of the endowment the kernel of the heterogeneously ambiguity-averse economy will be more (resp. less) elastic and, thus, steeper (resp. less steep) than that of the homogeneously ambiguity-averse economy.

To understand the implication for H-J bounds, consider two situations, a and b, with $\hat{m}_b > \hat{m}_a$. We interpret situation b (good times) as one where a typical consumer views the immediate future more optimistically relative to situation a (bad times). Then, the result shows that the H-J bound is *counter-cyclical* if there is heterogeneity in the relative ambiguity aversion, whereas it is *constant* across the cycle if relative ambiguity aversion is homogeneous. The H-J bound equals the highest Sharpe ratio that can be achieved

[24]The details of the argument can be found in Proposition 11 in Online-Appendix G.

by any portfolio of assets. Our result is empirically compelling since the Sharpe ratio for U.S. aggregate stock market is significantly counter-cyclical and volatile.[25]

In Assumption 1 the volatility is held constant across models. If we relaxed this assumption, we can show numerically that the pricing kernel might have an upward sloping segment.[26] This kind of non-monotonicity, anticipated in (Gollier 2011), provides an explanation of the so-called *pricing kernel puzzle,* discussed in (Hens & Reichlin 2013) and (Cuesdeanu & Jackwerth 2018). The puzzle refers to the empirical evidence that the downward slope of the pricing kernel implied by a risk averse EU representative consumer is violated in reality: there is an interval, away from extreme returns, where the pricing kernel is increasing, as in Figure 5 of (Rosenberg & Engle 2002) and our Figure 2. We illustrate this using our running example.

Running Example cont'd. *Let $\mathbf{P}_B$ and $\mathbf{P}_b$ be two log-normals such that $\mathbf{P}_B$ has a high mean and low variance, while $\mathbf{P}_b$ has a low mean and a high variance. The specification has the feature that the conditional likelihood of the bust model may* increase *given an increase in x. Since this regime is associated with a lower expected utility, this could lead to an increase of the first part of the weighted average $\left[\frac{p_b(x)}{p^\mu(x)}\phi'\left(E^{\mathbf{P}_b}u\left(\bar{X}\right)\right)+\frac{p_B(x)}{p^\mu(x)}\phi'\left(E^{\mathbf{P}_B}u\left(\bar{X}\right)\right)\right]u'(x)$ that appears in (13). On the other hand, an increase in x means a lower second component, that is, a lower marginal utility $u'(x)$. Thus, risk aversion and ambiguity aversion drive the kernel in opposite directions. If ambiguity aversion dominates for a range of endowment, then the state price instead of falling with an increase in x turns up, giving the pricing kernel a (locally)*

[25](Rosenberg & Engle 2002) obtain a measure of "empirical risk aversion" using the risk aversion implied by the pricing kernel they estimate. They show that this risk aversion varies counter-cyclically, supporting earlier results of (Fama & French 1989) who showed that risk premia are negatively correlated with the business cycle. See also (Lettau & Ludvigson 2010).

[26]Subsequent to our study, (Spengemann 2025) has investigated such a case in a fully Gaussian environment and obtained analytical results.

positive slope, as in Figure 2.[27]

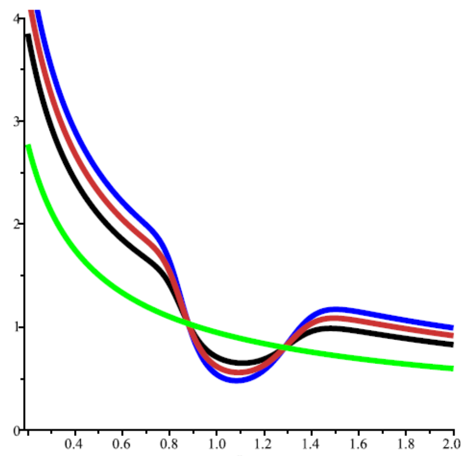

Figure 2: Graphs of the pricing kernels in this example for four economies. u is CRRA with $\alpha = 2/3$. (1) — : representative vNM consumer economy. (2) — : representative smooth ambiguity consumer economy, v is CRRA, $\gamma = 12$. (3) — : representative smooth ambiguity consumer economy, v is CRRA, $\gamma = 6$. (4) — : economy constituted by 2 consumers, one of whom has preferences as in (2) and the other as in (3).

4 Concluding remarks

We studied efficient uncertainty sharing in settings that incorporate ambiguity, in particular an ambiguous aggregate endowment and heterogeneous ambiguity attitudes, under the assumption of identifiable model uncertainty. This assumption, we argued, fits macro-finance environments characterized by regime-switching. We found that efficient allocations constitute a subset of model-by-model efficient allocations that provide expected-utility smoothing across models. This results in a smooth ambiguity representative consumer, if and only if the same condition that yields linear sharing rules in expected-utility settings holds. The impact of ambiguity aversion on the sharing rule

[27]The parameter values are estimates based on historical data, see Appendix.

then becomes explicit, particularly in how the slope of the rule changes contingent on models to allow less ambiguity averse consumers to take on a larger share of the uncertain endowment at better models. Efficient allocations are not comonotone but display other features such as expected-utility comonotonicity. Furthermore, the representative consumer displays decreasing relative ambiguity aversion. These insights continue to hold under the weaker notion of partial/set-identifiability.

One of the significant implications of our analysis is that the theoretical pricing kernel is potentially much closer to what is observed in reality. This suggests, perhaps, that existing financial markets already incorporate mechanisms for hedging ambiguous model uncertainty, even if not all necessary instruments and institutions for complete uncertainty sharing are currently available. Understanding how existing financial instruments accommodate model uncertainty, and whether new instruments are needed, remains an important avenue for future research.

Proofs

Proofs of Propositions in Section 1

Proof of Proposition 1 (Part 2.) At a Pareto optimum, if $\bar{X}^{\mathbf{P}}(\omega) = \bar{X}^{\mathbf{P}}(\omega')$ for some $\omega, \omega' \in \Omega_{\mathbf{P}}$ then $X_i^{\mathbf{P}}(\omega) = X_i^{\mathbf{P}}(\omega')$ for all i (by strict concavity of u_i). Hence, we can write $X_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = X_i^{\mathbf{P}}(\omega)$ for any $x \in \bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})$ and $\omega \in \Omega_{\mathbf{P}}$ with $\bar{X}^{\mathbf{P}}(\omega) = x$. Since $\bar{X}$ is unambiguous, $\mathbf{P}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = \mathbf{Q}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right) \equiv \zeta(x)$ for all $x \in \bar{X}(\Omega)$.

(only if) Let $Y = (Y^{\mathbf{P}})_{\mathbf{P}}$ be an efficient allocation (hence conditionally efficient). Assume there exist $i, x \in \bar{X}(\Omega)$ and $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ s.th. $Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) \neq Y_i^{\mathbf{Q}}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right)$. Define $Y_i^\star : \Omega \to \mathbb{R}_+$ so that: $Y_i^\star((\bar{X})^{-1}(x)) = \sum_{\mathbf{P}\in\mathcal{P}} \mu(P) Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)$. $Y^\star$ is feasible:

$$\sum_i Y_i^\star((\bar{X})^{-1}(x)) \;=\; \sum_i \sum_{\mathbf{P}} \mu(\mathbf{P}) Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = \sum_{\mathbf{P}} \mu(\mathbf{P}) x = x$$

Next, we show that $Y^\star$ Pareto dominates the allocation Y.

$$\begin{aligned}
U_i(Y_i) \quad &= \quad \sum_{\mathbf{P}} \mu(\mathbf{P}) \phi_i\left(E^{\mathbf{P}} u_i\left(Y_i^{\mathbf{P}}\right)\right) = \sum_{\mathbf{P}} \mu(\mathbf{P}) \phi_i\Big(\sum_{\omega\in\Omega_{\mathbf{P}}} \mathbf{P}(\omega) u_i\left(Y_i^{\mathbf{P}}(\omega)\right)\Big) \\
&= \quad \sum_{\mathbf{P}} \mu(\mathbf{P}) \phi_i\Big(\sum_{x\in\bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})} \mathbf{P}((\bar{X}^{\mathbf{P}})^{-1}(x)) u_i\left(Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\right)\Big) \\
\leq (<) &\quad \phi_i\Big(\sum_{\mathbf{P}} \mu(\mathbf{P}) \sum_{x\in\bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})} \zeta(x) u_i\left(Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\right)\Big) \\
\leq (<) &\quad \phi_i\Big(\sum_{x\in\bar{X}(\Omega)} \zeta(x) u_i\Big(\sum_{\mathbf{P}} \mu(\mathbf{P}) Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\Big)\Big) \\
&= \quad \phi_i\Big(\sum_{x\in\bar{X}(\Omega)} \zeta(x) u_i\left(Y_i^\star\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\right)\Big) = U_i(Y_i^\star)
\end{aligned}$$

Note, some weak inequalities in the derivation above are strict for at least one i. Hence, $Y^\star$ Pareto dominates Y, a contradiction.

(if) Let $Y = (Y^{\mathbf{P}})_{\mathbf{P}}$ be a conditionally efficient allocation such that $Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = Y_i^{\mathbf{Q}}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right)$ for all i, all $\mathbf{P}$, $\mathbf{Q}$ and all $x \in \bar{X}(\Omega)$. Assume it is not efficient. Then, there exists an efficient allocation $\hat{Y} = (\hat{Y}^{\mathbf{P}})_{\mathbf{P}}$

that Pareto dominates it. By the same argument as in the (only if) part of the proof, $\hat{Y}_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = \hat{Y}_i^{\mathbf{Q}}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right)$ for all i, all $\mathbf{P}$, $\mathbf{Q}$ and all $x \in \bar{X}(\Omega)$. Since endowment is unambiguous, we have that $E^{\mathbf{P}}u_i(\hat{Y}_i^{\mathbf{P}}) = E^{\mathbf{Q}}u_i(\hat{Y}_i^{\mathbf{P}})$ for all $i, \mathbf{P}, \mathbf{Q}$.
Therefore, $\sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right) = \phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right)$ for some (any) $\mathbf{P}$. The same holds for Y and hence, $\sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\left(E^{\mathbf{P}}(u_i(Y_i^{\mathbf{P}}))\right) = \phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right)$ for any $\mathbf{P}$. That $\hat{Y}$ Pareto dominates Y therefore means that $\phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right) \geq \phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right)$ for all i with a strict inequality for at least one. But this is a contradiction to the fact that Y is conditionally efficient. □

Proof of Proposition 2 Recall the necessary and sufficient condition for efficiency of an interior allocation $(X_i)_{i=1,\ldots,I}$, (4). Since the range of $\bar{X}$ is model-independent $\bar{X}(\Omega) = \bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})$ for all $\mathbf{P}$. For all $x \in \bar{X}(\Omega)$ and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$:

$$\frac{\psi^{\mathbf{Q}}(\omega_{\mathbf{Q}})}{\psi^{\mathbf{P}}(\omega_{\mathbf{P}})} = \frac{\phi_i'\left(E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)\right)u_i'\left(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{\phi_i'\left(E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right)\right)u_i'\left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \quad \forall i. \tag{14}$$

Let κ be s.th.

$$\frac{\phi_\kappa'\left(E^{\mathbf{Q}}u_\kappa\left(X_\kappa^{\mathbf{Q}}\right)\right)}{\phi_\kappa'\left(E^{\mathbf{P}}u_\kappa\left(X_\kappa^{\mathbf{P}}\right)\right)} \geq \frac{\phi_i'\left(E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)\right)}{\phi_i'\left(E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right)\right)} \quad \forall i = 1\ldots, I.$$

To simplify exposition, let $\kappa = 1$. By (14), for all x and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$:

$$\frac{u_1'\left(X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_1'\left(X_1^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \leq \frac{u_i'\left(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_i'\left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \quad \forall i.$$

If $X_1^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})$, then the l.h.s. is strictly greater than one. Hence $X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for every i, a contradiction to $\sum_i X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) = x = \sum_i X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$. Hence, $X_1^{\mathbf{P}}(\omega_{\mathbf{P}}) \leq X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for every x, $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$.

Since $u_1' > 0$, $E^{\mathbf{P}}u_1\left(X_1^{\mathbf{P}}\right) \leq E^{\mathbf{P}}u_1\left(X_1^{\mathbf{Q}}\right)$. Since $X_1^{\mathbf{Q}}$ is strictly monotone in $\bar{X}$, $\mathbf{P} \circ \left(X_1^{\mathbf{Q}}\right)^{-1}$ is FOS dominated by $\mathbf{Q} \circ \left(X_1^{\mathbf{Q}}\right)^{-1}$. Thus, $E^{\mathbf{P}}u_1\left(X_1^{\mathbf{Q}}\right) \leq E^{\mathbf{Q}}u_1\left(X_1^{\mathbf{Q}}\right)$. Hence, $E^{\mathbf{P}}u_1\left(X_1^{\mathbf{P}}\right) \leq E^{\mathbf{Q}}u_1\left(X_1^{\mathbf{Q}}\right)$. Thus,

$$\frac{\phi_i'\left(E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)\right)}{\phi_i'\left(E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right)\right)} \leq \frac{\phi_1'\left(E^{\mathbf{Q}}u_1\left(X_1^{\mathbf{Q}}\right)\right)}{\phi_1'\left(E^{\mathbf{P}}u_1\left(X_1^{\mathbf{P}}\right)\right)} \leq 1 \quad \forall i = 1, \ldots, I. \tag{15}$$

Since $\frac{u_1'\left(X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_1'\left(X_1^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \leq 1$ for every x and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$, by (14) and (15),

$$\frac{\psi^{\mathbf{Q}}(\omega_{\mathbf{Q}})}{\psi^{\mathbf{P}}(\omega_{\mathbf{P}})} = \frac{\phi_1'\left(E^{\mathbf{Q}}u_1\left(X_1^{\mathbf{Q}}\right)\right)u_1'\left(X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{\phi_1'\left(E^{\mathbf{P}}u_1\left(X_1^{\mathbf{P}}\right)\right)u_1'\left(X_1^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \leq 1. \tag{16}$$

To show that $E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right) \leq E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)$, consider two cases:

- If $X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) \leq X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for all x and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$, then we can show that $E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right) \leq E^{\mathbf{P}}u_i\left(X_i^{\mathbf{Q}}\right) \leq E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)$.
- If not, there are an x, an $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$, and an $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$ s.th. $X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$. For such an $(x, \omega_{\mathbf{P}}, \omega_{\mathbf{Q}})$, $\frac{u_i'\left(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_i'\left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} > 1$. By (16), $\frac{\phi_i'\left(E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)\right)}{\phi_i'\left(E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right)\right)} < 1$. Since $\phi_i'' < 0$, $E^{\mathbf{P}}u_i\left(X_i^{\mathbf{P}}\right) < E^{\mathbf{Q}}u_i\left(X_i^{\mathbf{Q}}\right)$.

□

Proofs of Propositions in Section 2

Proof of Lemma 1 Let $(X_i^{\mathbf{P}})_{\mathbf{P} \in \mathcal{P}, i=1,\ldots,I}$ be conditionally efficient. Focus first on the case where consumers are CARA with parameter α_i. The representative consumer is then CARA with parameter α s.th. $\sum_i (\alpha/\alpha_i) = 1$, and the sharing rule is $X_i = (\alpha/\alpha_i)\bar{X} + \tau_i$ where $\sum_i \tau_i = 0$. Direct computation yields

$$u_i^{-1}\left(E^{\mathbf{P}}u_i\left(X_i\right)\right) = \frac{\alpha}{\alpha_i}u^{-1}\left(E^{\mathbf{P}}u\left(\bar{X}\right)\right) + \tau_i. \tag{17}$$

Hence, $\sum_i u_i^{-1}\left(E^{\mathbf{P}} u_i(X_i)\right) = u^{-1}\left(E^{\mathbf{P}} u(\bar{X})\right)$.

Consider next the case where consumers have non-zero CMRT, $u_i(x_i) = \frac{\alpha}{1-\alpha}\left(\frac{x_i-\zeta_i}{\alpha}\right)^{1-\alpha}$ for $\alpha \neq 0$ and $\alpha \neq 1$.[28] The representative consumer has utility $u(x) = \frac{\alpha}{1-\alpha}\left(\frac{x-\zeta}{\alpha}\right)^{1-\alpha}$, where $\zeta = \sum_i \zeta_i$. The sharing rule is $X_i = \theta_i(\bar{X} - \zeta) + \zeta_i$ where $\sum_i \theta_i = 1$. Direct calculation yields $u_i^{-1}\left(E^{\mathbf{P}} u_i(X_i)\right) = \theta_i\left(u^{-1}\left(E^{\mathbf{P}} u(X)\right) - \zeta\right) + \zeta_i$ leading to $\sum_i u_i^{-1}\left(E^{\mathbf{P}} u_i\left(X_i\right)\right) = u^{-1}\left(E^{\mathbf{P}} u\left(\bar{X}\right)\right)$.

To prove the converse (for brevity, we focus on the non-zero CMRT case; the CARA case can be dealt with in a similar fashion), let (c_i) satisfies $\sum_i c_i = c_u^{\mathbf{P}}(\bar{X})$ and $c_i > \zeta_i$ for all i. Define $\theta_i = \frac{c_i-\zeta_i}{c_u^{\mathbf{P}}(\bar{X})-\zeta}$ and notice that $\sum_i \theta_i = 1$. Let $X_i^{\mathbf{P}} = \theta_i(\bar{X}^{\mathbf{P}} - \zeta) + \zeta_i$. $X_i^{\mathbf{P}}$ is feasible and, by Condition 1, conditionally efficient. Moreover, direct computation shows that $c_i^{\mathbf{P}}(X_i^{\mathbf{P}}) = c_i$ for all i. □

Proof of Proposition 3 Beyond the arguments in the text, it remains to prove that $[\exists i \text{ s.th. } \phi_i'' < 0] \Rightarrow \phi'' < 0$. Denote risk tolerance of u at x by $ART(x; u)$. Then, $ART(x, u) = \sum_i ART(g_i(x), u_i)$ where x is consumption levels and $g_i(x)$ a solution to (9) (Wilson 1968). Similarly, $ART(x, v) = \sum_i ART(f_i(x), v_i)$, where x represents certainty equivalents and $f_i(x)$ the solution to (10). Note $\sum_i g_i(x) = x = \sum_i f_i(x)$. As the u_i's are HARA with CMRT, if $ART(x_i, u_i) = a_i + bx$, then:

$$\sum_i ART(g_i(x), u_i) = \sum_i a_i + b\sum_i g_i(x) = \sum_i ART(f_i(x), u_i)$$

Since v_i is (strictly) more concave than u_i for all (at least one) i,

$$\sum_i a_i + \sum_i ART(f_i(x), u_i) > \sum_i a_i + \sum_i ART(f_i(x), v_i) = ART(x; v).$$

Hence, $ART(x; u) > ART(x; v)$ for all x, that is v is more concave than u, i.e., $\phi'' < 0$. EU-comonotonicity comes from comonotonicity of efficient allocations in vNM-economies applied to (10). □

[28]The HARA with CMRT family also includes $u_i(X_i) = \ln(X_i - \zeta_i)$.

Proof of Proposition 4 We give here an outline of the proof; the full argument can be found in Online-Appendix C. The proof is constructive and need at least four states. We focus here on the simplest possible economy (with precisely four states) where we can illustrate the construction, that applies more generally. Consider a two-consumer, four-state economy, $\Omega = \{\omega_{\mathbf{P}}, \omega'_{\mathbf{P}}, \omega_{\mathbf{Q}}, \omega'_{\mathbf{Q}}\}$, with $\mathcal{P} = \{\mathbf{P}, \mathbf{Q}\}$, $\Omega_{\mathbf{P}} = \{\omega_{\mathbf{P}}, \omega'_{\mathbf{P}}\}$ and $\Omega_{\mathbf{Q}} = \{\omega_{\mathbf{Q}}, \omega'_{\mathbf{Q}}\}$. Pick u_1 and u_2, two Bernoulli functions with linear risk tolerance *with different slopes*, thus violating Condition 1, and assume $\mathbf{P}(\omega_{\mathbf{P}}) > \mathbf{Q}(\omega_{\mathbf{Q}})$. Pick an endowment such that $\bar{X}(\omega_{\mathbf{P}}) = \bar{X}(\omega_{\mathbf{Q}}) > \bar{X}(\omega'_{\mathbf{P}}) = \bar{X}(\omega'_{\mathbf{Q}})$ and a conditionally-efficient allocation such that $X_1^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ and $X_1^{\mathbf{P}}(\omega'_{\mathbf{P}}) > X_1^{\mathbf{Q}}(\omega'_{\mathbf{Q}})$, and hence, $X_2^{\mathbf{P}}(\omega_{\mathbf{P}}) < X_2^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ and $X_2^{\mathbf{P}}(\omega'_{\mathbf{P}}) < X_2^{\mathbf{Q}}(\omega'_{\mathbf{Q}})$. Obviously, we get that $E^{\mathbf{P}} u_1(X_1^{\mathbf{P}}) > E^{\mathbf{Q}} u_1(X_1^{\mathbf{Q}})$. How about consumer 2? Choose the $\mathbf{P}$ and $\mathbf{Q}$-conditionally efficient allocations "close enough" such that $X_2^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_2^{\mathbf{Q}}(\omega'_{\mathbf{Q}})$. Then, if $\mathbf{P}(\omega_{\mathbf{P}})$ is sufficiently high and $\mathbf{Q}(\omega_{\mathbf{Q}})$ sufficiently low, we obtain that $E^{\mathbf{P}} u_2(X_2^{\mathbf{P}}) > E^{\mathbf{Q}} u_2(X_2^{\mathbf{Q}})$. Hence, the overall allocation is EU-comonotone and, consequently, efficient for some (ϕ_1, ϕ_2) –see (Hara et al. 2022). Recall that the risk tolerance of the representative consumer in a vNM-economy at an efficient allocation is the sum of individual consumers' (Wilson 1968). Thus, given that both have linear risk tolerance but with different slope, if 1 has higher marginal risk tolerance than 2, then the representative consumer's risk tolerance is lower under $\mathbf{P}$ than under $\mathbf{Q}$ (since contingent on $\mathbf{P}$, 1 (resp. 2) has more (resp. less) than under $\mathbf{Q}$, irrespective of the realization of endowment). Hence, the representative consumer's Bernoulli utility, u, is not just a function of the endowment but also of the model.This is incompatible with the overall value function V having the requisite smooth ambiguity form, $V(\bar{X}) = \mu(\mathbf{P})\phi(E^{\mathbf{P}} u(\bar{X}^{\mathbf{P}})) + \mu(\mathbf{Q})\phi(E^{\mathbf{Q}} u(\bar{X}^{\mathbf{Q}}))$. □

Proof of Proposition 5

1. Each $\mathbf{P}$-conditionally efficient allocation can be written $X_i^{\mathbf{P}} = \theta_i^{\mathbf{P}}(\bar{X} -$

$\zeta)+\zeta_i$ for some $\theta_i^{\mathbf{P}}$ (Section 3.6, (Back 2017)). We prove that $\theta_i^{\mathbf{P}}$ is a function of $c_u^{\mathbf{P}}$. Efficient allocations can be obtained by solving (10), for some $(\lambda_i^{\mathbf{P}})_i$, to allocate aggregate certainty equivalents $c^{\mathbf{P}}$ under each model $\mathbf{P}$.

For each $c > \zeta$, let $(\hat{f}_i(c))_i$ be the solution to (10). Then, $\hat{f}_i(c_u^{\mathbf{P}}) = u_i^{-1}\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right)$. For each $z > 0$, define $f_i(z) = \hat{f}_i(z+\zeta) - \zeta_i$ and $\theta_i(z) = f_i(z)/z$. Then $u_i^{-1}\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right) = \theta_i(c_u^{\mathbf{P}} - \zeta) \times (c_u^{\mathbf{P}} - \zeta) + \zeta_i$. Since v is the value function of (10), the envelope theorem implies that $\lambda_i v_i'(f_i(z)+\zeta_i) = v'(z+\zeta)$. Hence, $v'(z+\zeta) > 0$. Differentiating w.r.t. z, $\lambda_i v_i''(f_i(x)+\zeta_i) f_i'(z) = v''(z+\zeta)$. Dividing each side of the second equality by the corresponding side of the first :

$$-\frac{v_i''(f_i(z)+\zeta_i) f_i(z)}{v_i'(f_i(z)+\zeta_i)} \frac{f_i'(z)}{f_i(z)} + \frac{v''(z+\zeta)}{v'(z+\zeta)} = 0$$

for every i and $z > 0$. Since v_i exhibits HARA with parameters (γ_i, ζ_i),

$$\gamma_i \frac{f_i'(z)}{f_i(z)} + \frac{v''(z+\zeta)}{v'(z+\zeta)} = 0. \tag{18}$$

Since $\sum_i f_i(z) = z$ $\exists i$ s.th. $f_i'(z) > 0$. Thus, $v''(z+\zeta) < 0$. Hence, $f_i'(z) > 0$ for every i. Moreover,

$$\frac{d}{dz} \ln \frac{f_j(z)}{f_i(z)} = \frac{f_j'(z)}{f_j(z)} - \frac{f_i'(z)}{f_i(z)} = -\frac{v''(z+\zeta)}{v'(z+\zeta)} \left(\frac{1}{\gamma_j} - \frac{1}{\gamma_j}\right) \gtreqless 0$$

if and only if $\gamma_i \gtreqless \gamma_j$. Since $f_j(z)/f_i(z) = \theta_j(z)/\theta_i(z)$, the sign property in part 1 (a) is proved. Part 1 (b) follows from (21) below.

2. (Corollary 7 part 2 of (Hara, Huang & Kuzmics 2007)). One can write (18) as $\theta_i(z)/\gamma_i = f_i'(z)/b(z)$. Hence,

$$\sum_i \theta_i(z) \frac{1}{\gamma_i} = \frac{1}{b(z)}. \tag{19}$$

Differentiating w.r.t. z, $\sum_i \theta_i'(z) \frac{1}{\gamma_i} = -\frac{b'(z)}{(b(z))^2}$. Since $\sum_i \theta_i'(z) = 0$,

$$\sum_i \theta_i'(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)}\right) = -\frac{b'(z)}{(b(z))^2}. \tag{20}$$

Since

$$\theta_i'(z) = \frac{\theta_i(z)}{z}\left(\frac{b(z)}{\gamma_i} - 1\right), \tag{21}$$

we have

$$\sum_i \theta_i'(z)\left(\frac{1}{\gamma_i} - \frac{1}{b(z)}\right) = \frac{b(z)}{z}\sum_i \theta_i(z)\left(\frac{1}{\gamma_i} - \frac{1}{b(z)}\right)^2.$$

If $\min_i \gamma_i < \max_i \gamma_i$, by (19), $\exists i$ s.th. $1/\gamma_i < 1/b(z)$ (and another i s.th. $1/\gamma_i > 1/b(z)$). Thus, the r.h.s. is strictly positive. By (20), $b'(z) < 0$. If $\min_i \gamma_i = \max_i \gamma_i$, then, by (19), $1/\gamma_i = 1/b(z)$ $\forall i$. By (20), $b'(z) = 0$. □

Proofs of Proposition in Section 3

Lemma 2 establishes results used in the proof of Proposition 6.

Lemma 2. *1. Let P be any non-degenerate probability on $\mathbb{R}_{++}$. For $n = 1, 2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be continuous. Assume π_2 is non-increasing and π_2/π_1 is strictly increasing. Then, $\sigma(\pi_1)/E(\pi_1) > \sigma(\pi_2)/E(\pi_2)$, where E and σ are the mean and standard deviation under P.*

2. For $n = 1, 2$, let P_n be any non-degenerate probability on $\mathbb{R}_{++}$. Assume that P_n has a probability density function (pdf) g_n and that there is a $k > 1$ such that $g_1(x) = kg_2(kx)$ for every $x > 0$. Let $\pi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be differentiable. Assume that $\pi' < 0$ and $-\pi'(x)x/\pi(x)$ is strictly decreasing in x. Then $\sigma^{P_1}(\pi)/E^{P_1}(\pi) > \sigma^{P_2}(\pi)/E^{P_2}(\pi)$, where, for each n, E^{P_n} and σ^{P_n} are the mean and standard deviation under P_n.

Proof of Lemma 2

1. For each n, the integral of the function $x \mapsto (E(\pi_n))^{-1}\pi_n(x)$ under P is equal to one. Since it is continuous, (the graphs of) these two functions $n = 1, 2$ cross at least once. Since π_2/π_1 is strictly increasing, they cross

exactly once. Let x^* be such that $\pi_1(x^*)/E(\pi_1) = \pi_2(x^*)/E(\pi_2)$ and denote this value by z^*. Then $\pi_1(x)/E(\pi_1) \gtreqless \pi_2(x)/E(\pi_2)$ if and only if $x \lesseqgtr x^*$. Since π_2 is non-increasing,

$$\frac{\pi_1(x)}{E(\pi_1)} \gtreqless \frac{\pi_2(x)}{E(\pi_2)} \gtreqless z^* \text{ if and only if } x \lesseqgtr x^*.$$

Thus, for every $x \neq x^*$, $\left(\frac{\pi_1(x)}{E(\pi_1)} - z^*\right)^2 > \left(\frac{\pi_2(x)}{E(\pi_2)} - z^*\right)^2$. If $x = x^*$, then this inequality would hold as an equality. Since P is not degenerate,

$$\int \left(\frac{\pi_2(x)}{E(\pi_2)} - z^*\right)^2 P(dx) > \int \left(\frac{\pi_1(x)}{E(\pi_1)} - z^*\right)^2 P(dx).$$

Note that, for each $n = 1, 2$,

$$\begin{aligned}\frac{\sigma(\pi_n)^2}{E(\pi_n)^2} &= \int \left(\frac{\pi_n(x)}{E(\pi_n)} - 1\right)^2 P(dx) \\ &= \int \left(\frac{\pi_n(x)}{E(\pi_n)} - z^*\right)^2 P(dx) - (z^* - 1)^2.\end{aligned}$$

Thus, $\sigma(\pi_1)^2/E(\pi_1)^2 > \sigma(\pi_2)^2/E(\pi_2)^2$. Thus, $\sigma(\pi_1)/E(\pi_1) > \sigma(\pi_2)/E(\pi_2)$.

2. We prove this part by applying part 1. To do so, write P for P_1, g for g_1, and π_1 for π. Define π_2 by letting $\pi_2(x) = \pi_1(kx)$ for every $x \in I$. We now show that $E^P(\pi_2) = E^{P_2}(\pi)$ and $\sigma^P(\pi_2) = \sigma^{P_2}(\pi)$. Since $g_1(x) = kg_2(kx)$, the change-of-variable formula implies that

$$E^P(\pi_2) = \int \pi_2(x)g_1(x)dx = \int \pi_2(kx)kg_2(kx)dx = \int \pi(x)f_2(x)dx = E^{P_2}(\pi).$$

By this equality and the change-of-variables formula,

$$\begin{aligned}\sigma^P(\pi_2) &= \left(\int \left(\pi_2(x) - E^P(\pi_2)\right)^2 g_1(x)dx\right)^{1/2} \\ &= \left(\int \left(\pi(x) - E^{P_2}(\pi)\right)^2 f_2(x)dx\right)^{1/2} = \sigma^{P_2}(\pi).\end{aligned}$$

Thus, $\sigma^P(\pi_2)/E^P(\pi_2) = \sigma^{P_2}(\pi)/E^{P_2}(\pi)$. It is thus enough to show $\sigma^P(\pi_1)/E^P(\pi_1) > \sigma^P(\pi_2)/E^P(\pi_2)$. By part 1, it suffices to prove that π_2/π_1 is strictly increasing. Differentiate both sides of $\pi_2(x) = \pi_1(kx)$ with respect to x, we obtain $\pi_2'(x) = \pi_1'(kx)k$. Thus, $-\frac{\pi_2'(x)x}{\pi_2(x)} = -\frac{\pi_1'(kx)kx}{\pi_1(kx)}$. Since $k > 1$ and $-\pi_1'(x)x/\pi_1(x)$ is a strictly decreasing function of x, $-\frac{\pi_1'(kx)kx}{\pi_1(kx)} < -\frac{\pi_1'(x)x}{\pi_1(x)}$. Thus, $-\pi_2'(x)x/\pi_2(x) < -\pi_1'(x)x/\pi_1(x)$, that is, $-\pi_2'(x)/\pi_2(x) < -\pi_1'(x)/\pi_1(x)$ for every x. This is equivalent to $(\pi_2/\pi_1)' > 0$, thus completing the proof.

□

We now proceed to prove Proposition 6. It is convenient for this proof to proceed to a change of variable, as it were. Recall, that $\bar{X}$ is log-normally distributed. Let $s \equiv \log(x)$ for a generic element $x \in \mathbb{R}$ and $S = \log(\bar{X}(\Omega)) = \mathbb{R}$. s is thus normally distributed. Recall that Ω is identified with $\bar{X}(\Omega) \times \mathcal{P}$, which is, in turn, identified with $\mathbb{R} \times \mathbb{R}$. Thus, $\mathbf{P}$ is a joint distribution over $(s, m) \in \mathbb{R} \times \mathbb{R}$. Denote the pdfs of the second-order belief $\mathcal{N}(\hat{m}, \hat{\sigma}^2)$ and a first-order belief $\mathcal{N}(m, \sigma^2)$ by p_M and $p_{S|M}(\,\cdot\,|\,m)$, respectively. It follows from Bayes' formula that the conditional second-order belief given s is

$$\mathcal{N}\left(\frac{\hat{\sigma}^2 s + \sigma^2 \hat{m}}{\sigma^2 + \hat{\sigma}^2}, \frac{\hat{\sigma}^2 \sigma^2}{\sigma^2 + \hat{\sigma}^2}\right). \tag{22}$$

Denote its pdf by $p_{M|S}(\,\cdot\,|\,s)$. Observe we may write the kernel (13) as

$$\pi_{u,\phi}(s) = u'(\exp(s))h(s, \mu), \tag{23}$$

to identify the component h which encapsulates the effect of ambiguity aversion,

$$h(s, \mu) \equiv \int_{\mathcal{P}} \frac{p(s)}{p^{\mu}(s)} \phi'\left(E^{\mathbf{P}} u\left(\bar{X}^{\mathbf{P}}\right)\right) \mu(d\mathbf{P}). \tag{24}$$

Thus, in the case of interest here, (24) can be rewritten as

$$h(s, \hat{m}) = \int \frac{v'(c(m))}{u'(c(m))} p_{M|S}(m\,|\,s)dm, \tag{25}$$

where $c(m) = u^{-1}\left(E^m u(\bar{X})\right)$ and E^m is the expectation under $\mathcal{N}(m, \sigma^2)$. The relation (23) can be rewritten as $\pi_{u,\phi}(s, \hat{m}) = \lambda(\hat{m})u'(\exp X(s))h(s, \hat{m})$.

Write $r = \hat{\sigma}^2/(\sigma^2 + \hat{\sigma}^2)$, then $0 < r < 1$. Denote by q the pdf of $\mathcal{N}\left(0, \frac{\hat{\sigma}^2\sigma^2}{\sigma^2+\hat{\sigma}^2}\right)$. Then, the pdf of (22) coincides with the function $s \mapsto q(m - (rs + (1-r)\hat{\sigma}))$ and (25) can be rewritten as

$$h(s, \hat{m}) = \int_{-\infty}^{\infty} \frac{v'(c(m))}{u'(c(m))} q(m - (rs + (1-r)\hat{m}))\, dm.$$

The following two lemmas are consequences of Proposition 10 in Appendix F, which is a general result on strict log-supermodularity (SLSPM for short).

Lemma 3. *Suppose that Assumption 1 holds, that u exhibits CRRA, and that the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x. Then, h is strictly log-supermodular, that is,*

$$h(s_1, \hat{m}_1)h(s_2, \hat{m}_2) < h(\max\{s_1, s_2\}, \max\{\hat{m}_1, \hat{m}_2\})h(\min\{s_1, s_2\}, \min\{\hat{m}_1, \hat{m}_2\})$$

for all $(s_1, \hat{m}_1)$ and $(s_2, \hat{m}_2)$, unless $(s_1, \hat{m}_1) \leq (s_2, \hat{m}_2)$ or $(s_1, \hat{m}_1) \geq (s_2, \hat{m}_2)$.

Proof of Lemma 3 By part 1 of Assumption 1,

$$c(m) = \exp\left(m + \frac{\sigma^2}{2}(1-\alpha)\right).$$

Define $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{++}$ by

$$f(s, \hat{m}, m) = \frac{v'(c(m + rs + (1-r)\hat{m}))}{u'(c(m + rs + (1-r)\hat{m}))} q(m).$$

Since $c'(m + rs) = c(m + rs)$,

$$\begin{aligned}
\frac{\partial}{\partial s} \ln f(s, \hat{m}, m) &= \frac{\mathrm{d}}{\mathrm{d}s} \ln v'(c(m + rs + (1-r)\hat{m}))) - \frac{\mathrm{d}}{\mathrm{d}s} \ln u'(c(m + rs + (1-r)\hat{m}))) \\
&= \frac{v''(c(m + rs + (1-r)\hat{m})))}{v'(c(m + rs + (1-r)\hat{m})))} c'(m + rs + (1-r)\hat{m}))r - \\
&\quad - \frac{u''(c(m + rs + (1-r)\hat{m})))}{u'(c(m + rs + (1-r)\hat{m})))} c'(m + rs + (1-r)\hat{m}))r \\
&= \left(\frac{v''(c(m + rs + (1-r)\hat{m})))c(m + rs + (1-r)\hat{m}))}{v'(c(m + rs + (1-r)\hat{m})))} - \alpha\right) r. \qquad (26)
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial \hat{m}} \ln f(s, \hat{m}, m) = \left(\frac{v''(c(m + rs + (1-r)\hat{m}))c(m + rs + (1-r)\hat{m}))}{v'(c(m + rs + (1-r)\hat{m}))} - \alpha \right) (1-r).$$

Thus,

$$\frac{\partial^2}{\partial s \partial \hat{m}} \ln f(s, \hat{m}, m) = \left. \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)} \right|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m}) r(1-r) > 0,$$

since v has differentiably strictly decreasing relative risk aversion. Similarly,

$$\frac{\partial^2}{\partial s \partial m} \ln f(s, \hat{m}, m) = \left. \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)} \right|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m}) r > 0,$$

$$\frac{\partial^2}{\partial \hat{m} \partial m} \ln f(s, \hat{m}, m) = \left. \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)} \right|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m})(1-r) > 0.$$

Thus, by Proposition 10, the function $(s, \hat{m}) \mapsto \int_{-\infty}^{\infty} f(s, \hat{m}, m) \, \mathrm{d}m$ has SLSPM. By the change of variable,

$$\int_{-\infty}^{\infty} f(s, \hat{m}, m) \, \mathrm{d}m = \int_{-\infty}^{\infty} \frac{v'(c(m))}{u'(c(m))} b(m - (rs + (1-r)\hat{m})) \, \mathrm{d}m = h(s, \hat{m}). \tag{27}$$

This completes the proof. □

Lemma 4. *Suppose that Assumption 1 holds, that u exhibits CRRA, and that the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x. Then, for every $\hat{m} \in \mathbb{R}$,*

$$\frac{\frac{\partial h}{\partial s}(s, \hat{m})}{h(s, \hat{m})}$$

is strictly increasing in $s \in \mathbb{R}$.

Proof of Lemma 4 Let $\hat{m} \in \mathbb{R}$. Let f be as in the proof of Lemma 3.

Define $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{++}$ by $k(s, \varepsilon, m) = f(s + \varepsilon, \hat{m}, m)$. By (26),

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial \varepsilon} \ln k(s, \varepsilon, m) &= \frac{\partial^2}{\partial s^2} \ln f(s, \hat{m}, m) \\
&= \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m})r^2 > 0, \\
\frac{\partial^2}{\partial m \partial \varepsilon} \ln k(s, \varepsilon, m) &= \frac{\partial^2}{\partial m \partial s} \ln k(s, \varepsilon, m) \quad = \quad \frac{\partial^2}{\partial s \partial m} \ln f(s, \hat{m}, m) \\
&= \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m})r > 0.
\end{aligned}$$

By Proposition 10, the function $(s, \varepsilon) \mapsto \int_{-\infty}^{\infty} k(s, \varepsilon, m)\, \mathrm{d}m$ has SLSPM. Since $k(s, \varepsilon, m) = f(s + \varepsilon, \hat{m}, m)$, by (27), this function is equal to $(s, \varepsilon) \mapsto h(s + \varepsilon, \hat{m})$. Since it has SLSPM, if $s_1 < s_2$ and $\varepsilon > 0$, then $\frac{h(s_1+\varepsilon,\hat{m})}{h(s_1,\hat{m})} < \frac{h(s_2+\varepsilon,\hat{m})}{h(_2,\hat{m})}$. This means that $h(s + \varepsilon, \hat{m})/h(s, \hat{m})$ is a strictly increasing function of s. Since

$$\frac{d}{ds} \ln \frac{h(s + \varepsilon, \hat{m})}{h(s, \hat{m})} = \frac{\frac{\partial h}{\partial s}(s + \varepsilon, \hat{m})}{h(s + \varepsilon, \hat{m})} - \frac{\frac{\partial h}{\partial s}(s, \hat{m})}{h(s, \hat{m})},$$

and the left-hand side is nonnegative, $\frac{\partial h}{\partial s}(s, \hat{m})/h(s, \hat{m})$ is non-decreasing in s. To prove that it is, in fact, strictly increasing, suppose not. Then, there is an interval, say $(\underline{s}, \overline{s})$, over which it is constant. Take a small $\varepsilon > 0$. Then, over an interval of s with $\underline{s} < s < s + \varepsilon < \overline{s}$, the right-hand side is constantly equal to 0. Hence, $h(s+\varepsilon, \hat{m})/h(s, \hat{m})$ is constant. But this is a contradiction. Thus, $\frac{\partial h}{\partial s}(s, \hat{m})/h(s, \hat{m})$ is strictly increasing in s. □

Proof of Proposition 6 1. Both results follow from direct calculation.
2. By differentiating the logarithm of (23) and multiplying by -1, we obtain

$$-\frac{\pi'_{u,\phi}(s)}{\pi_{u,\phi}(s)} = -\frac{u''(\exp(s))\exp(s)}{u'(\exp(s))} - \frac{\frac{\partial h}{\partial s}(s, \mu)}{h(s, \mu)} \quad \text{for every } s.$$

$$\varepsilon(x; \pi_{u,\phi}) = -\frac{\pi'_{u,\phi}(x)x}{\pi_{u,\phi}(x)} = -\frac{u''(x)x}{u'(x)} - \frac{\frac{\partial h}{\partial s}(\ln x, \mu)}{h(\ln x, \mu)} \tag{28}$$

for every $x > 0$. Since u exhibits constant relative risk aversion, the first fraction on the right-hand side of (28) (where μ is replaced by $\hat{m}$) is independent of x. By Lemma 4, the second fraction is strictly increasing in x. Thus, $\varepsilon(x; \pi_{u,\phi})$ is strictly decreasing in x.

Let $\hat{m}_b > \hat{m}_a$. By Lemma 3, $h(s, \hat{m}_b)/h(s, \hat{m}_a)$ is strictly increasing in s. Thus, by (23), where μ is replaced by $\hat{m}_a$ and $\hat{m}_b$, $\pi_{u,\phi}(x; \hat{m}_b)/\pi_{u,\phi}(x; \hat{m}_a)$ is strictly increasing in x. Thus, by part 1 of Lemma 2,

$$\frac{\sigma^{\hat{m}_a}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_a)\right)}{E^{\hat{m}_a}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_{a1})\right)} > \frac{\sigma^{\hat{m}_a}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_b)\right)}{E^{\hat{m}_a}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_b)\right)}. \tag{29}$$

For each n, under the second-order belief $\mathcal{N}(\hat{m}_n, \hat{\sigma}^2)$, the reduced probability over S coincides with $\mathcal{N}(\hat{m}_n, \sigma^2 + \hat{\sigma}^2)$. Since $\bar{X}(s) = \exp(s)$, the reduced probability over consumption levels coincides with the log-normal distribution $\mathcal{LN}(\hat{m}_n, \hat{\sigma}^2 + \sigma^2)$. Let g_n be the pdf of this distribution and $k = \exp(\hat{m}_b - \hat{m}_a)$, then $k > 1$ and $g_1(x) = kg_2(kx)$ for every $x > 0$. Thus, by part 2 of Lemma 2,

$$\frac{\sigma^{\hat{m}_a}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_b)\right)}{E^{\hat{m}_a}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_b)\right)} > \frac{\sigma^{\hat{m}_b}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_b)\right)}{E^{\hat{m}_b}\left(\pi_{u,\phi}(\,\cdot\,; \hat{m}_b)\right)}. \tag{30}$$

By (29) and (30), the proof is completed. □

Parameters for Figure 2

x is lognormal with mean and volatility parameters m and σ, that are unknown. Consumers put probability 1/2 on (m_1, σ_1) and 1/2 on (m_2, σ_2). For the parameter values, we rely on the two-regime (annualized) specification in Table 6 of (Gadea, Gómez-Loscos & Pérez-Quirós 2020). (Gadea, Gómez-Loscos & Pérez-Quirós 2020) divided the time span of data (from 1875 to 2014) into two historical regimes. We assume a recession partially identifies the distributions as a set of two possible distributions because consumers think the recessionary distributions in either historical regime is possible. Analogously, an expansion also partially identifies a set of two distributions. Following the argument in Section 2.3, consumers behave as if the worst distribution in each partially-identified set is in operation. Thus, we obtain $(m_1, \sigma_1) = (.04, .011)$ ($\mathbf{P}_B$), and $(m_2, \sigma_2) = (-.15, .11)$ ($\mathbf{P}_b$). The four economies considered are:

Homogeneous and ambiguity-neutral : EU representative consumer with a CRRA utility function with relative risk aversion equal to 2/3.

Homogeneous and ambiguity-averse : smooth ambiguity representative consumer with CRRA u with relative risk aversion equal to 2/3 and CRRA v with index 12.

Homogeneous and ambiguity-averse : smooth ambiguity representative consumer with CRRA u with relative risk aversion equal to 2/3 and CRRA v with index 6.

Heterogeneous and ambiguity-averse There is a consumer of each of the two ambiguity-averse types described above, with equal weight.

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Online-Appendix

A Ambiguity aversion and revealed beliefs

Denote by $b(\Omega_{\mathbf{P}})$ a bet that pays c^* on $\Omega_{\mathbf{P}}$ and c_* off it, and by $b(\Omega \setminus \Omega_{\mathbf{P}})$ the bet on the complementary event $\Omega \setminus \Omega_{\mathbf{P}}$. Note that $\mathbf{P}(\Omega_{\mathbf{P}}) = 1$. Normalize u_i and ϕ_i so that $u_i(c_*) = 0$ and $\phi_i(0) = 0$, and write $h = u(c^*)$. Consumer i evaluates these bets as: $U_i(b(\Omega_{\mathbf{P}})) = \mu(\mathbf{P})\,\phi_i(\mathbf{P}(\Omega_{\mathbf{P}})\,h) = \mu(\mathbf{P})\,\phi_i(h)$ and $U_i(b(\Omega \setminus \Omega_{\mathbf{P}})) = (1-\mu(\mathbf{P}))\,\phi_i(h)$. Consider next a lottery ℓ^{π} which pays c^* with a probability π and c_* with probability $1-\pi$. Then, $U_i(\ell^{\pi}) = \phi_i(\pi h)$. If ϕ_i is strictly concave, then $U_i(b(\Omega_{\mathbf{P}})) < U_i(\ell^{\mu(\mathbf{P})})$ and $U_i(b(\Omega \setminus \Omega_{\mathbf{P}})) < U_i(\ell^{1-\mu(\mathbf{P})})$. Define $\underline{\pi}, \bar{\pi} \in [0,1]$ so that $U_i(\ell^{\underline{\pi}}) = U_i(b(\Omega_{\mathbf{P}}))$ and $U_i(\ell^{1-\bar{\pi}}) = U_i(b(\Omega \setminus \Omega_{\mathbf{P}}))$. Since ϕ is strictly increasing, $\underline{\pi} < \mu(\mathbf{P}) < \bar{\pi}$. Moreover, $\underline{\pi}$ satisfies

$$\begin{aligned} \phi_i(\underline{\pi}h + (1-\underline{\pi})0) &= \mu(\mathbf{P})\,\phi_i(h) + (1-\mu(\mathbf{P}))\phi_i(0) \\ \Leftrightarrow \underline{\pi}h &= \phi_i^{-1}(\mu(\mathbf{P})\,\phi_i(h)) \end{aligned}$$

Applying a quadratic approximation, we get, letting λ_{ϕ_i} be the Arrow-Pratt measure of absolute risk aversion for the function ϕ_i (see Online-Appendix B for further detail).

$$\begin{aligned} \underline{\pi}h &= \mu(\mathbf{P})\,h - \frac{\lambda_{\phi_i}(0)}{2}\left[\mu(\mathbf{P})\,h^2 - (\mu(\mathbf{P})\,h)^2\right] + o\left(h^2\right) \\ \Leftrightarrow \underline{\pi} &= \mu(\mathbf{P}) - \frac{\lambda_{\phi_i}(0)}{2}\mu(\mathbf{P})\,(1-\mu(\mathbf{P}))\,h + o(h) \end{aligned}$$

Similarly, $\bar{\pi} = \mu(\mathbf{P}) + \frac{\lambda_{\phi_i}(0)}{2}\mu(\mathbf{P})\,(1-\mu(\mathbf{P}))\,h + o(h)$. Hence, the "probability matching" interval for $\Omega_{\mathbf{P}}$ is given by $[\underline{\pi}, \bar{\pi}]$. Its length is increasing in λ_{ϕ_i}.

B Relative ambiguity aversion

We relate the measure of relative ambiguity aversion introduced in Section 2.2 to ambiguity premiums (see also (Cerreia-Vioglio, Maccheroni & Marinacci 2022)). Let h be a random variable defined on Ω and w be the initial consumption level. Denote by $\mathbf{P}^{\mu}$ the reduced measure $\int_{\mathcal{P}} \mathbf{Q}\mu(d\mathbf{Q})$, and by λ_u the Arrow-Pratt measure of absolute risk aversion for a Bernoulli utility u. The variance $(\sigma^\mu)^2 (E^\cdot(h))$ of the function $E^\cdot(h) : \mathbf{P} \mapsto E^{\mathbf{P}}(h)$ under μ reflects the uncertainty on the expected values and encapsulates ambiguity. The certainty equivalent for a proportional ambiguous prospect xh can be approximated as[29]

$$\begin{aligned} C(x+xh) &= x + E^{\mathbf{P}^\mu}(xh) - \frac{x^2}{2}\lambda_u(x)(\sigma^{\mathbf{P}^\mu})^2(h) \\ &\quad - \frac{x^2}{2}(\lambda_v(x) - \lambda_u(x))(\sigma^\mu)^2\left(E^{\mathbf{P}^\mu}(h)\right) + o\left(\|h\|^2\right) \end{aligned}$$

Since $\phi = v \circ u^{-1}$, $\lambda_\phi(u(x)) = \frac{1}{u'(x)}(\lambda_v(x) - \lambda_u(x))$, that is, $\lambda_\phi(u(x))u'(x) = \lambda_v(x) - \lambda_u(x)$. The ambiguity premium for xh is obtained by subtracting the risk premium from the overall uncertainty premium and, as a proportion of wealth, equal to

$$((\lambda_v(x) - \lambda_u(x))\,x) \times \frac{1}{2}(\sigma^\mu)^2\left(E^{\mathbf{P}^\mu}(h)\right) = \lambda_\phi(u(x))u'(x)\,x \times \frac{1}{2}(\sigma^\mu)^2\left(E^{\mathbf{P}^\mu}(h)\right).$$

In our HARA specification, it is convenient to express the ambiguity premium in terms of the effective consumption $x - \zeta$. By differentiating $v = \phi \circ u$, we obtain

$$-\frac{v''(x)}{v'(x)} = -\frac{\phi''(u(x))}{\phi'(u(x))}u'(x) - \frac{u''(x)}{u'(x)}. \tag{31}$$

By multiplying both sides by $x - \zeta$, we obtain, under Condition 2:

$$-\frac{\phi''(u(x))}{\phi'(u(x))}u'(x)(x-\zeta) = \gamma - \alpha. \tag{32}$$

C Proof of Proposition 4.

For the purpose of this Appendix, denote the value function of (9) by $u(x, \lambda)$. Then, u is the representative consumer's (inner) Bernoulli utility function, where dependence on the vector λ of utility weights is made explicit. Similarly, denote the value function of (3) by $V^{\mathbf{P}}(\bar{X}^{\mathbf{P}}, \lambda)$ and the value function of (2) by $V\left((\bar{X}^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}, \lambda\right)$. Then,

$$V\left((\bar{Y}^{\mathbf{P}})_{\mathbf{P}}, \lambda\right) = \sum_{\mathbf{P}} \mu(\mathbf{P}) V^P\left(\bar{Y}^{\mathbf{P}}, \lambda\right) \tag{33}$$

[29]This is akin to the quadratic approximation of certainty equivalent obtained by (Maccheroni, Marinacci & Ruffino 2013)

for every $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$.

Denote the solution to (9) by $(f_i(x,\lambda))_i$. Then $(f_i)_i$ is the risk-sharing rule, with the dependence on the vector λ of utility weights made explicit. By the envelope theorem,

$$\frac{\partial u}{\partial x}(x,\lambda) = \lambda_i u_i'(f_i(x,\lambda)) \tag{34}$$

for every i. Denote the risk tolerances of u_i and u by t_i and t. (Wilson 1968) showed that $t(x,\lambda) = \sum_i t_i(f_i(x,\lambda))$ for every (x,λ). Hence,

$$\nabla_\lambda t(x,\lambda) = \sum_i t_i'(f_i(x,\lambda))\nabla_\lambda f_i(x,\lambda). \tag{35}$$

Lemma 5. $\nabla_\lambda t(x,\lambda) = 0$ *if and only if* $t_1'(f_1(x,\lambda)) = \cdots = t_I'(f_I(x,\lambda))$.

Proof of Lemma 5 Although this lemma is true for an arbitrary I, we give a proof only for $I=2$ to save space. By (34), $\lambda_1 u_1'(f_1(x,\lambda)) = \lambda_2 u_2'(f_2(x,\lambda))$. By differentiating both sides w.r.t. λ_1:

$$u_1'(f_1(x,\lambda)) + \lambda_1 u_1''(f_1(x,\lambda))\frac{\partial f_1}{\partial \lambda_1}(x,\lambda) = \lambda_2 u_2''(f_2(x,\lambda))\frac{\partial f_2}{\partial \lambda_1}(x,\lambda).$$

Since $\sum_i (\partial f_i/\partial \lambda_1)(x,\lambda) = 0$, $u_1'(f_1(x,\lambda)) = -\frac{\partial f_1}{\partial \lambda_1}(x,\lambda)\sum_i \lambda_i u_i''(f_i(x,\lambda))$. Hence, $(\partial f_1/\partial \lambda_1)(x,\lambda) > 0$. Thus,

$$\frac{\partial t}{\partial \lambda_1}(x,\lambda) = \left(t_1'(f_1(x,\lambda)) - t_2'(f_2(x,\lambda))\right)\frac{\partial f_1}{\partial \lambda_1}(x,\lambda) = 0$$

if and only if $t_1'(f_1(x,\lambda)) = t_2'(f_2(x,\lambda))$. □

If $(X_i^{\mathbf{P}})_i$ is a solution to (3), then, by the envelope theorem,

$$\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}(\omega)} = \lambda_i \phi_i'\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right) u_i'(X_i^{\mathbf{P}}(\omega))\mathbf{P}(\omega) \text{ for all } i \text{ and } \omega. \tag{36}$$

Lemma 6. *For each* $\mathbf{P} \in \mathcal{P}$*, let* $(X_i^{\mathbf{P}})_i$ *be a solution to* (3)*. Write* $\lambda_i^{\mathbf{P}} = \lambda_i \phi_i'\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right)$ *and* $\lambda^{\mathbf{P}} = (\lambda_i^{\mathbf{P}})_i$*. Suppose that there is a pair of a differentiable function* $u : \mathbb{X} \to \mathbb{R}$ *and a differentiable function* $\phi : u(\mathbb{X}) \to \mathbb{R}$ *such that* $V^{\mathbf{P}}(\bar{Y},\lambda) = \phi(E^{\mathbf{P}} u(\bar{Y}))$ *for all* $\mathbf{P} \in \mathcal{P}$ *and* $\bar{Y} : \Omega \to \mathbb{X}$*. Then, for all* ω_1 *and* $\omega_2 \in \Omega_{\mathbf{P}}$*,*

$$\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}$$

depends only on the values of $\bar{X}^{\mathbf{P}}(\omega_1)$ *and* $\bar{X}^{\mathbf{P}}(\omega_2)$*, that is, if* $\bar{X}^{\mathbf{P}}(\omega_1) = \bar{X}^{\mathbf{Q}}(\omega_3)$ *and* $\bar{X}^{\mathbf{P}}(\omega_2) = \bar{X}^{\mathbf{Q}}(\omega_4)$*, then* $\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})} = \int_{\bar{X}^{\mathbf{Q}}(\omega_3)}^{\bar{X}^{\mathbf{Q}}(\omega_4)} \frac{dx}{t(x,\lambda^{\mathbf{Q}})}$ *for all* $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$*,* $\omega_1, \omega_2 \in \Omega_{\mathbf{P}}$*, and* $\omega_3, \omega_4 \in \Omega_{\mathbf{Q}}$*.*

Proof of Lemma 6 First, we prove that

$$\frac{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \exp\left(-\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}\right).$$

Indeed, by (36),

$$\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega)}}{\mathbf{P}(\omega)} = \lambda_i^{\mathbf{P}} u_i'(X_i^{\mathbf{P}}(\omega))$$

for every ω. Thus, the right-hand side is independent of i. Hence, the first-order condition for a solution to (9) is met, and $X_i^{\mathbf{P}}(\omega) = f_i(\bar{X}^{\mathbf{P}}(\omega), \lambda^{\mathbf{P}})$ for all i and $\omega \in \Omega$. Thus, by (34), $\lambda_i^{\mathbf{P}} u_i'(X_i^{\mathbf{P}}(\omega)) = \frac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega), \lambda^{\mathbf{P}})$. Hence,

$$\frac{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \frac{\frac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega_2), \lambda^{\mathbf{P}})}{\frac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega_1), \lambda^{\mathbf{P}})} = \exp\left(-\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}\right). \tag{37}$$

On the other hand, by assumption, the chain rule implies that

$$\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega)} = \phi'(E^{\mathbf{P}} u(\bar{X})) u'(\bar{X}^{\mathbf{P}}(\omega)) \mathbf{P}(\omega)$$

for every ω. Thus,

$$\frac{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \frac{u'(\bar{X}^{\mathbf{P}}(\omega_2))}{u'(\bar{X}^{\mathbf{P}}(\omega_1))}$$

for all ω_1 and ω_2. Since the right-hand side depends only on the values of $\bar{X}^{\mathbf{P}}(\omega_1)$ and $\bar{X}^{\mathbf{P}}(\omega_2)$, so is the left-hand side. The lemma follows now from (37). □

Lemma 7. *Suppose that there is a pair of a differentiable function $u : \mathbb{X} \to \mathbb{R}$ and a differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ such that $V((\bar{Y}^{\mathbf{P}})_{\mathbf{P}}, \lambda) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi(E^{\mathbf{P}}u(\bar{Y}^{\mathbf{P}}))$ for every $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$, where $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$ for every $\mathbf{P}$. Then, $V^{\mathbf{P}}(\bar{Y}, \lambda) = \phi(E^{\mathbf{P}}u(\bar{Y}))$ for all $\mathbf{P}$ and $\bar{Y} : \Omega \to \mathbb{X}$.*

Proof of Lemma 7 Let $\mathbf{Q} \in \mathcal{P}$, and $(\bar{X}^{\mathbf{P}})_{\mathbf{P}}$ and $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$ be two endowments such that $\bar{X}^{\mathbf{P}} = \bar{Y}^{\mathbf{P}}$ for every $\mathbf{P} \in \mathcal{P} \setminus \{\mathbf{Q}\}$. By assumption and (33),

$$\phi(E^{\mathbf{Q}}u(\bar{X}^{\mathbf{Q}})) - \phi(E^{\mathbf{Q}}u(\bar{Y}^{\mathbf{Q}})) = V^{\mathbf{Q}}\left(\bar{X}^{\mathbf{Q}}, \lambda\right) - V^{\mathbf{Q}}\left(\bar{Y}^{\mathbf{Q}}, \lambda\right).$$

Therefore, for every $\mathbf{P} \in \mathcal{P}$, there is an $a^{\mathbf{P}} \in \mathbb{R}$ such that $V^{\mathbf{P}}(\bar{Y}^{\mathbf{P}}) = \phi(E^{\mathbf{P}}u(\bar{Y}^{\mathbf{P}})) + a^{\mathbf{P}}$ for every $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$. Hence, $\sum_{\mathbf{P}} \mu(\mathbf{P})a^{\mathbf{P}} = 0$. Let $\bar{Y} : \Omega \to \mathbb{X}$ be (deterministic) endowments for which there is an $x \in \mathbb{X}$ such that $\bar{Y}(\omega) = x$ for every ω. Then, for every $\mathbf{P}$, the solution $(Y_i^{\mathbf{P}})_i$ to (2) is given by letting $Y_i^{\mathbf{P}}$ be the deterministic consumption x_i such that $\lambda_i\phi_i'(u_i(x_i))u_i'(x_i)$ is independent of i, and $V^{\mathbf{P}}(\bar{Y}) = \sum_i \lambda_i\phi_i(u_i(x_i))$. Thus, whenever $\bar{Y}$ is deterministic, $V^{\mathbf{P}}(\bar{Y})$ is independent of $\mathbf{P}$. Hence, $a^{\mathbf{P}}$ is independent of $\mathbf{P}$. Thus, $a^{\mathbf{P}} = 0$ for every $\mathbf{P}$. Hence, $V^{\mathbf{P}}(\bar{Y}) = \phi(E^{\mathbf{P}}u(\bar{Y}))$ for all $\mathbf{P}$ and $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$. □

Proposition 7. *Assume $|\Omega| \geq 4$. For each i, let u_i be a (inner) Bernoulli utility function with the following property: for each i, there is an $x_i^* \in \mathbb{X}_i$ such that it is not true that $t_1'(x_1^*) = t_2'(x_2^*) = \cdots = t_I'(x_I^*)$. Then, there are: for each i, a Bernoulli utility function ϕ_i over expected utility levels; a (common) second-order belief μ on Ω; endowments $\bar{X} : \Omega \to \mathbb{X}$ whose range* *is model-independent; and a vector $\lambda^\star$ of utility weights, such that such that if V is defined by (2), then there is no pair of a (inner) Bernoulli utility function u and a Bernoulli utility function ϕ over expected utility levels such that $V\left((\bar{Y}^{\mathbf{P}})_{\mathbf{P}}, \lambda\right) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi\left(E^{\mathbf{P}}u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$.*

Proof of Proposition 7 Suppose that for each i, there is an $x_i^* \in \mathbb{X}_i$ such that it is not true that $t_1'(x_1^*) = t_2'(x_2^*) = \cdots = t_I'(x_I^*)$. For each i, let $\lambda_i^* = (u_i'(x_i^*))^{-1}$, and $\lambda^* = (\lambda_i^*)_i$. Write $x^* = \sum_i x_i^*$. Then, $x_i^* = f_i(x^*, \lambda^*)$ for every i. By Lemma 5, $\nabla_\lambda t(x^*, \lambda^*)$ is a nonzero vector. Thus, there is a $\kappa \in \mathbb{R}^I$ such that $\nabla_\lambda t(x^*, \lambda^*)\kappa > 0$. Note here that

$$D_\lambda u_i(f_i(x^*, \lambda^*)) = u_i'(f_i(x^*, \lambda^*))\nabla_\lambda f_i(x^*, \lambda^*) \in \mathbb{R}^I.$$

Let $\delta > 0$ be so large that $D_\lambda u_i(f_i(x^*, \lambda^*))\kappa + \delta > 0$ for every i, then there is a neighborhood $\mathbb{Y}$ of x^* and a neighborhood Λ of λ^* such that $D_\lambda u_i(f_i(x, \lambda))\kappa + \delta > 0$ and $\nabla_\lambda t(x, \lambda)\kappa > 0$ for all i and $(x, \lambda) \in \mathbb{Y} \times \Lambda$. Then,

$$\frac{d}{d\varepsilon} t(x, \lambda^* + \varepsilon\kappa) = \nabla_\lambda t(x, \lambda^* + \varepsilon\kappa)\kappa > 0$$

for every $x \in \mathbb{Y}$ and every ε sufficiently close to 0. Hence, for every $x \in \mathbb{Y}$, $t(x, \lambda^* + \varepsilon\kappa)$ is a strictly increasing function of ε around 0.

Since $\Omega \geq 4$, there is a partition $(\Xi^1, \Xi^2, \Xi^3, \Xi^4)$ of Ω where each Ξ^n is non-empty. Let $x^1, x^2 \in \mathbb{X}$ be such that $x^1 < x^2$. Define $\bar{X} : \Omega \to \mathbb{X}$ by

$$\bar{X}(\omega) = \begin{cases} x^1 & \text{if } \omega \in \Xi^1 \cup \Xi^3 \\ x^2 & \text{if } \omega \in \Xi^2 \cup \Xi^4 \end{cases}$$

Define $\rho > 0$ so that $(u_i(f_i(x^2, \lambda)) - u_i(f_i(x^1, \lambda)))\rho > \delta$ for all i. Let $\mathbf{P}^0 \in \Delta(\Omega)$ be s. th. $\mathbf{P}^0(\omega) > 0$ for all $\omega \in \Xi^1 \cup \Xi^2$ and $\mathbf{P}^0(\omega) = 0$ for all $\omega \in \Xi^3 \cup \Xi^4$. For each $\varepsilon > 0$ sufficiently close to 0, let $\mathbf{P}^\varepsilon \in \Delta(\Omega)$ be s. th.

$$\mathbf{P}^\varepsilon(\omega) = \begin{cases} \frac{1}{|\Xi^3|}(\mathbf{P}^0(\Xi^1) - \varepsilon\rho) & \text{if } \omega \in \Xi^3, \\ \frac{1}{|\Xi^4|}(\mathbf{P}^0(\Xi^2) + \varepsilon\rho) & \text{if } \omega \in \Xi^4, \\ 0 & \text{if } \omega \in \Xi^1 \cup \Xi^2 \end{cases}$$

Then, $\mathbf{P}^\varepsilon(\Xi^3) = \mathbf{P}^0(\Xi^1) - \varepsilon\rho$, $\mathbf{P}^\varepsilon(\Xi^4) = \mathbf{P}^0(\Xi^2) + \varepsilon\rho$ and $\mathbf{P}^\varepsilon(\Xi^1 \cup \Xi^2) = 0$. Fix a sufficiently small $\varepsilon^\star > 0$ and let $\mathcal{P} = \{\mathbf{P}^0, \mathbf{P}^{\varepsilon^\star}\}$. Then, $\mathcal{P}$ is point-identified with kernel k s.th.

$$k(\omega) = \begin{cases} \mathbf{P}^0 & \text{if } \omega \in \Xi^1 \cup \Xi^2 \\ \mathbf{P}^{\varepsilon^\star} & \text{if } \omega \in \Xi^3 \cup \Xi^4 \end{cases}$$

Moreover, $\Omega_{\mathbf{P}^0} = \Xi^1 \cup \Xi^2$ and $\Omega_{\mathbf{P}^{\varepsilon^\star}} = \Xi^3 \cup \Xi^4$. Thus, the range of $\bar{X}$ is model independent. Let μ be a second-order belief s.th. $\mu(\mathbf{P}^0) > 0$ and $\mu(\mathbf{P}^{\varepsilon^\star}) > 0$. Then, by definition of δ and ρ, $\forall \varepsilon > 0$:

$$\begin{aligned}\frac{d}{d\varepsilon} E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)) &= (u_i(f_i(x^2, \lambda^* + \varepsilon\kappa))) - u_i(f_i(x^1, \lambda^* + \varepsilon\kappa))\rho \\ &\quad + \sum_{\omega\in\Omega} \mathbf{P}^\varepsilon(\omega) D_\lambda u_i(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))\kappa \\ &> \delta + \sum_{\omega\in\Omega} \mathbf{P}^\varepsilon(\omega)(-\delta) = 0.\end{aligned}$$

Thus, by Proposition 10 of (Hara et al. 2022) for each i, there is a twice continuously differentiable ϕ_i with $\phi_i'' \leq 0 < \phi_i'$ such that $\left(\left(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)\right)_i\right)_{\varepsilon=0,\varepsilon^\star}$ is an efficient allocation of the economy $\left((u_i, \phi_i, \mu)_i, \bar{X}\right)$.

Since $\left(\left(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)\right)_i\right)_{\varepsilon=0,\varepsilon^\star}$ is an efficient allocation of the economy $\left((u_i, \phi_i, \mu)_i, \bar{X}\right)$, there is a $\nu \in \mathbb{R}^I_{++}$ such that it is a solution to (2) when λ is replaced by ν. The first-order condition is that for all ε and ω,

$$\nu_i \phi_i' \left(E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa))\right) u_i'(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))$$

is independent of i. Write $\lambda_i^{\mathbf{P}^\varepsilon} = \nu_i \phi_i' \left(E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa))\right)$. By definition,

$$(\lambda_i^* + \varepsilon\kappa_i) u_i'(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))$$

is independent of i. Thus, $\lambda_i^{\mathbf{P}^\varepsilon}/(\lambda_i^* + \varepsilon\kappa_i)$ is independent of i. Denote it by c^ε. Then $\lambda^{\mathbf{P}^\varepsilon} = c^\varepsilon(\lambda^* + \varepsilon\kappa)$. Hence, $u\left(\cdot, \lambda^{\mathbf{P}^\varepsilon}\right) = c^\varepsilon u\left(\cdot, \lambda^* + \varepsilon\kappa\right)$. Thus, $t\left(\cdot, \lambda^{\mathbf{P}^\varepsilon}\right) = t\left(\cdot, \lambda^* + \varepsilon\kappa\right)$. Hence,

$$\int_{x^1}^{x^2} \frac{dx}{t(x, \lambda^{\mathbf{P}^\varepsilon})} = \int_{x^1}^{x^2} \frac{dx}{t(x, \lambda^* + \varepsilon\kappa)}. \tag{38}$$

Since $t(x, \lambda^* + \varepsilon\kappa)$ is a strictly increasing function of ε for every x, each side of this equality is a strictly decreasing function of ε. In particular, each side is greater for $\varepsilon = 0$ than for $\varepsilon = \varepsilon^\star$.

Suppose that there is a pair of a twice continuously differentiable function $u : \mathbb{X} \to \mathbb{R}$ satisfying $u'' < 0 < u'$ and a twice continuously differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ satisfying $\phi'' \leq 0 < \phi'$ such that $V(\left(\bar{Y}^{\mathbf{P}}\right)_{\mathbf{P}}) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$, where V is the value function of (2). Then, by Lemma 7, for every $\mathbf{P} \in \mathcal{P}$, $V^{\mathbf{P}}(\bar{Y}^{\mathbf{P}}) = \phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $\bar{Y}^{\mathbf{P}}$:

$\Omega \to \mathbb{X}$, where $V^{\mathbf{P}}$ is the value function of (3). Thus, by Lemma 6, the left-hand side of (38) is independent of ε. In particular, it takes the same value for $\varepsilon = 0$ and $\varepsilon = \varepsilon^{\star}$. This is a contradiction. Hence, there is no pair of a differentiable function $u : \mathbb{X} \to \mathbb{R}$ and a differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ such that $V(\left(\bar{Y}^{\mathbf{P}}\right)_{\mathbf{P}}) = \sum_{\mathbf{P}} \mu(\mathbf{P}) \phi \left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$. □

D Constant absolute risk aversion

We study here an economy where u_i and v_i are HARA with zero marginal risk tolerance.[30]

Assumption 2. *Assume u_i is CARA with risk aversion $\alpha_i > 0$ and v_i is CARA with risk aversion $\gamma_i \geq \alpha_i$*

Assumption 2 is equivalent to assume u_i and v_i are HARA with CMRT (with parameters $(0, \frac{1}{\alpha_i})$ and $(0, \frac{1}{\gamma_i})$ respectively). Let $\phi_i = v_i \circ u_i^{-1}$, so $\phi_i(t) \propto -(-t^{\gamma_i/\alpha_i})$. Hence, our economy consists of smooth ambiguity-averse consumers with heterogeneous risk aversion and heterogeneous ambiguity aversion, parameterized by CARA Bernoulli utilities with risk aversion coefficient $\alpha_i > 0$ and by a power function with index $\frac{\gamma_i}{\alpha_i} \geq 1$, respectively.

Proposition 8. *Let $(X_i^{\mathbf{P}})_{\mathbf{P},i}$ be an efficient allocation of an economy that satisfies Assumption 2. Let $\alpha = \left(\sum_i \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_i \gamma_i^{-1}\right)^{-1}$. Then,*

1. *For each P, there are constants $(\tau_i^{\mathbf{P}})_{i=1,\ldots,I}$ s.th. $\sum_i \tau_i^{\mathbf{P}} = 0$ and $X_i^{\mathbf{P}} = (\alpha/\alpha_i)\bar{X} + \tau_i^{\mathbf{P}}$ for every i.*

2. *For every i, there is a function $\tau_i : (-\infty, \infty) \to (-\infty, \infty)$ and constants κ_i such that $\tau_i(c) = \frac{\gamma}{\gamma_i}\left(1 - \frac{\gamma_i/\alpha_i}{\gamma/\alpha}\right) c + \kappa_i$ with $\sum_i \kappa_i = 0$ and*

$$\tau_i^{\mathbf{P}} = \tau_i(c^{\mathbf{P}}) \tag{39}$$

   with $c^{\mathbf{P}} = u^{-1}(E^{\mathbf{P}} u(\bar{X}))$, where u, the representative consumer's utility function, is CARA with absolute risk aversion coefficient α.

3. *In the smooth ambiguity representative consumer's utility $\phi(t) \propto -(-t^{\gamma/\alpha})$ and $v = \phi \circ u$ is CARA with parameter γ.*

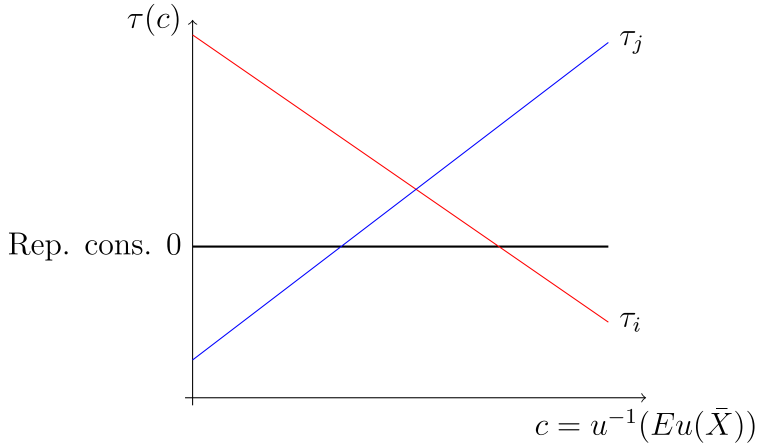

Figure 3: Constant risk tolerance case. The Figure shows the transfers as a function of the certainty equivalents for two consumers, i and j. Consumer i is more ambiguity-averse than, and j is less ambiguity-averse than, the representative consumer.

As P varies, the efficient allocation rule adjusts by varying the intercept term of the linear sharing rule, $\tau_i^{\mathbf{P}}$, a term denoting transfers that sum to zero across all the consumers. The function $\tau_i^{\mathbf{P}}$ is itself linear in the aggregate certainty equivalent. Figure (3) gives a graphical depiction showing how $\tau_i^{\mathbf{P}}$ varies as a function of the representative consumer's certainty equivalent for two consumers in this economy as established in Proposition 8.

If ambiguity attitudes were homogeneous, i.e., $\gamma_i/\alpha_i = \gamma_j/\alpha_j$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected-utility consumers: for all i, $\tau_i^{\mathbf{P}}$ is independent of $\mathbf{P}$.

E Non-zero marginal risk tolerance

We provide here a complement to Proposition 5 and give the limit behavior of $\theta_i(.)$ and b.

Proposition 9. *Consider the functions θ_i and RAA_ϕ constructed in Proposition 5. Then,*

[30]While this class of utility functions is usually not the one considered in the DSGE literature, it admits an easy representation for the efficient allocations and the representative consumer's utility function, while allowing for heterogeneity.

1. *$\theta_i(z) \to 0$ as $z \to 0$ if $\gamma_i \neq \max_{i=1,\ldots,I} \gamma_i$ and $\theta_i(z) \to 0$ as $z \to \infty$ if $\gamma_i \neq \min_{i=1,\ldots,I} \gamma_i$.*

2. *$RAA_\phi(z) \to \max_{i=1,\ldots,I} \gamma_i - \alpha$ as $z \to 0$, and $RAA_\phi(z) \to \min_{i=1,\ldots,I} \gamma_i - \alpha$ as $z \to \infty$.*

Proof of Proposition 9 The l.h.s. of (18) is equal to the derivative of the logarithm of the function $z \mapsto (f_i(z))^{\gamma_i} v'(z+\zeta)$. Hence this function is, in fact, constant. Thus, if there were an i s.th. $f_i(z)$ is bounded from above, then $v'(z)$ would be bounded away from zero. Then, $f_i(z)$ would be bounded from above for every i. This would contradict the assumption that $\sum_i f_i(z) = z$ for every $z > 0$. Hence, for every i, $f_i(z) \to \infty$ as $z \to \infty$. We can analogously show that for every i, $f_i(z) \to 0$ as $z \to 0$. This also shows that $v'(x) \to \infty$ as $x \to \zeta$ and $v'(x) \to 0$ as $x \to \infty$.

Denote the constant value of $(f_i(z))^{\gamma_i} v'(z+\zeta)$ by κ_i. Then, for every i and j,

$$0 < \theta_i(z) = \frac{f_i(z)}{z} < \frac{f_i(z)}{f_j(z)} = \frac{\left(\dfrac{\kappa_i}{v'(z+\zeta)}\right)^{1/\gamma_i}}{\left(\dfrac{\kappa_j}{v'(z+\zeta)}\right)^{1/\gamma_j}} = \frac{\kappa_i^{1/\gamma_i}}{\kappa_j^{1/\gamma_j}} \left(v'(z+\zeta)\right)^{1/\gamma_j - 1/\gamma_i}.$$

If $\gamma_i < \max_{i=1,\ldots,I} \gamma_i = \gamma_j$, then $1/\gamma_j - 1/\gamma_i < 0$. Since $v'(z+\zeta) \to \infty$ as $z \to 0$, the far right-hand side of the above equality converges to 0 as $z \to 0$. Hence $\theta_i(z) \to 0$ as $z \to 0$. We can analogously show that for every i, if $\gamma_i > \min_{i=1,\ldots,I} \gamma_i$, then $\theta_i(z) \to 0$ as $z \to \infty$. The limiting behavior of RAA_ϕ follows. □

We now explain the qualitative features of the graph of the shares θ_i as a function of the aggregate certainty equivalent.

Part 1(b) of Proposition 5 implies that, as we move from worse to better models, a consumer whose relative ambiguity aversion is greater (smaller) than that of the representative consumer around $c^{\mathbf{P}}$ will see their share decrease (resp. increase) for models with certainty equivalents marginally greater than $c^{\mathbf{P}}$ as shown in Figure 4.

Consider consumer I with the largest relative ambiguity aversion in the economy. By part 2 of Proposition 5, their relative ambiguity aversion is greater than that of the representative consumer (at all $c^{\mathbf{P}}$). By part 1(b)

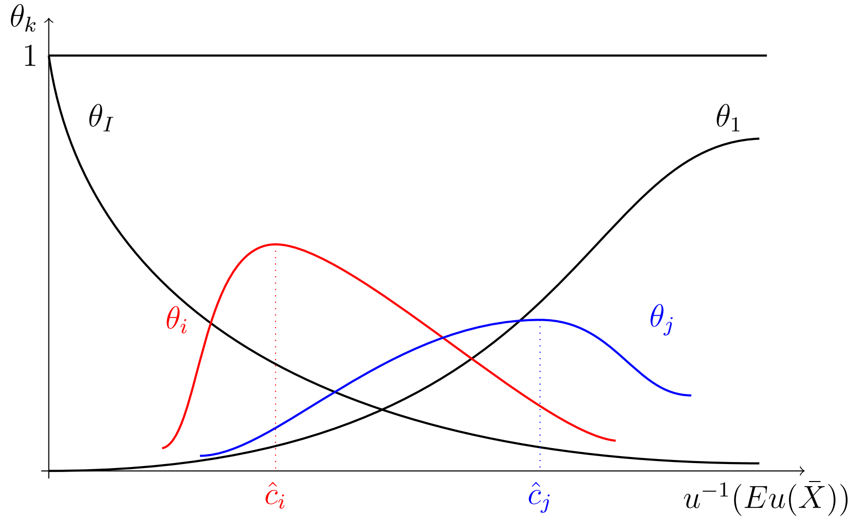

Figure 4: Comparing consumption shares θ_k under Condition 2. Consumer I (resp. 1) is the most (resp. the least) relatively ambiguity-averse. i is more relatively ambiguity-averse than consumer j.

of Proposition 5, θ_I will be negatively sloped everywhere. Analogously, consumer 1, with the lowest relative ambiguity aversion in the economy, will have a θ_1 that is positively sloped everywhere. From 1 of Proposition 9, the most relatively ambiguity-averse consumers get all of $\bar{X} - \zeta$ at the worst models. Therefore, at these models the representative consumer's relative ambiguity aversion is $\max_{i=1,\ldots,I} \gamma_i - \alpha$. Hence, by part 1(b) of Proposition 5, any consumer i with relative ambiguity aversion less than $\max_{i=1,\ldots,I} \gamma_i - \alpha$ will have their share increasing at least initially. Since the representative consumer has decreasing relative ambiguity aversion, we will reach a model, identified by $\hat{c}_i$ in Figure 4, where the representative consumer's relative ambiguity aversion falls below i's; hence, i's share is decreasing to the right of $\hat{c}_i$. For a consumer j relatively less ambiguity-averse than i, the representative consumer's ambiguity aversion has to decrease further before j's share peaks. Hence, $\hat{c}_j$ is to the right of $\hat{c}_i$. Taken together, the most relatively ambiguity-averse consumers get protected with extra shares at the worst models, the "middling" relative ambiguity-averse consumers get extra shares at the "middling" models and the least relatively ambiguity-averse ones get compensated by extra shares at the best models.

Finally, note that if $\gamma_i - \alpha = \gamma_j - \alpha$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected-utility consumers: for all i, θ_i is a constant function.

F Strict log-supermodularity

In this Appendix, we give a general result on strict log-supermodularity (SLSPM for short) from which part 2 of Proposition 6 can be derived.

Let N be a positive integer. For each $x = (x_n)_{n=1,2,\dots,N} \in \mathbb{R}^N$ and each $y = (y_n)_{n=1,2,\dots,N} \in \mathbb{R}^N$, we write $x \geq y$ when $x_n \geq y_n$ for every n. We also write $x \vee y = (\max\{x_n, y_n\})_{n=1,2,\dots,N}$ and $x \wedge y = (\min\{x_n, y_n\})_{n=1,2,\dots,N}$. For each $x = (x_n)_{n=1,2,\dots,N} \in \mathbb{R}^N$, we write $x_{-N} = (x_n)_{n=1,2,\dots,N-1} \in \mathbb{R}^{N-1}$. By a slight abuse of notation, we use $\geq$, $\leq$, $\vee$, and $\wedge$ for vectors in $\mathbb{R}^{N-1}$ as well.

Let $f : \mathbb{R}^N \to \mathbb{R}_+$. We say that f is *strictly log-supermodular* (SLSPM for short) if

$$f(x)f(y) < f(x \vee y)f(x \wedge y)$$

for every $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ unless $x \leq y$ or $x \geq y$. That is, the strict log-supermodularity is a stronger property than the log-supermodularity (LSPM) in that the left-hand side is strictly smaller than the right-hand side. If $x \leq y$ or $x \geq y$, then $\{x, y\} = \{x \vee y, x \wedge y\}$ and the left- and right-hand sides would necessarily be equal. The constraint that neither should hold is needed to exclude this case. If $f(x) > 0$ for every $x \in \mathbb{R}^N$, then f is SLSPM if and only if $\ln f$ is strictly supermodular in the sense of Topkis (1998, Section 2.6.1).

Throughout this Appendix, we assume, for every $f : \mathbb{R}^N \to \mathbb{R}_+$ under consideration, that f is differentiable and $f(x) > 0$ for every $x \in \mathbb{R}^N$.

The first part of the following result is stated in Topkis (1998, Section 2.6.1). The second part can be proved in an analogues manner. The proof is omitted.

Lemma 8. *1. f is LSPM if and only if, for all n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a nondecreasing function of x_m.*

2. f is SLSPM if, for every n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a strictly increasing function of x_m.

Proposition 10. *Suppose that for all $m < N$ and n, $\partial \ln f(x)/\partial x_m$ is non-decreasing in x_n, and strictly increasing in x_n if $n = N$. Define $g : \mathbb{R}^{N-1} \to \mathbb{R}_{++}$ by $g(x_{-N}) = \int_{\mathbb{R}} f(x_{-N}, x_N)\, \mathrm{d}x_N$ for every $x_{-N} \in \mathbb{R}^{N-1}$. Then g is SLSPM.*

The assumptions of this proposition imply that f is LSPM but not that f is SLSPM. In fact, they can be met even when f is not SLSPM. The proposition, thus, implies that g can be SLSPM even when f is not. For a twice continuously differentiable f, they are satisfied if, for every $x \in \mathbb{R}^N$, $\frac{\partial^2}{\partial x_m \partial x_N} \ln f(x) > 0$ for every $m < N$, and $\frac{\partial^2}{\partial x_m \partial x_n} \ln f(x) \geq 0$for all $m < N$ and $n \neq m$.

The following proof method is essentially due to Karlin and Rinott (1980, Theorem 2.1). We only need to take special care of preserving strict inequalities under integration.

Proof of Proposition 10 By Fubini's theorem,

$$
\begin{aligned}
& g(x_{-N})g(y_{-N}) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_{-N}, z) f(y_{-N}, w) \, \mathrm{d}w \mathrm{d}z = \int_{\mathbb{R}\times\mathbb{R}} f(x_{-N}, z) f(y_{-N}, w) \, \mathrm{d}(z, w) \\
&= \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z=w\}} f(x_{-N}, z) f(y_{-N}, w) \, \mathrm{d}(z, w) \\
&\quad + \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z<w\}} (f(x_{-N}, z) f(y_{-N}, w) + f(y_{-N}, w) f(x_{-N}, z)) \, \mathrm{d}(z, w).
\end{aligned} \tag{40}
$$

We can similarly show that

$$
\begin{aligned}
& g(x_{-N} \vee y_{-N}) g(x_{-N} \wedge y_{-N}) \\
&= \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z=w\}} f(x_{-N} \vee y_{-N}, z) f(x_{-N} \wedge y_{-N}, w) \, \mathrm{d}(z, w) \\
&\quad + \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z<w\}} (f(x_{-N} \vee y_{-N}, z) f(y_{-N} \wedge y_{-N}, w) \\
&\qquad\qquad + f(x_{-N} \vee y_{-N}, w) f(x_{-N} \wedge y_{-N}, z)) \, \mathrm{d}(z, w).
\end{aligned} \tag{41}
$$

When $z = w$, $(x_{-N}, z) \vee (y_{-N}, w) = (x_{-N} \vee y_{-N}, z)$ and $(x_{-N}, z) \wedge (y_{-N}, w) = (x_{-N} \wedge y_{-N}, w)$. Since f is LSPM,

$$
f(x_{-N}, z) f(y_{-N}, w) \leq f(x_{-N} \vee y_{-N}, z) f(x_{-N} \wedge y_{-N}, w).
$$

Thus, the first term of the right-hand side of (40) is less than or equal to that of (41). To compare the second terms, assume that $z < w$ and that it is false that $x_{-N} \leq y_{-N}$. Write

$$A(z,w) = f(x_{-N},z)f(y_{-N},w), \qquad C(z,w) = f(x_{-N} \vee y_{-N},z)f(y_{-N} \wedge y_{-N},w),$$
$$B(z,w) = f(x_{-N},w)f(y_{-N},z), \qquad D(z,w) = f(x_{-N} \vee y_{-N},w)f(x_{-N} \wedge y_{-N},z).$$

Note first that

$$\begin{aligned} A(z,w)B(z,w) &= (f(x_{-N},z)f(y_{-N},z))\,(f(x_{-N},w)f(y_{-N},w)) \\ &\leq (f(x_{-N} \vee y_{-N},z)f(x_{-N} \wedge y_{-N},z))\,(f(x_{-N} \vee y_{-N},w)f(y_{-N} \wedge y_{-N},w)) \\ &= C(z,w)D(z,w). \end{aligned}$$

Next, without loss of generality, we can assume that there is an M with $1 \leq M < N$ s.th. $x_n > y_n$ if and only if $n \leq M$. Then,

$$\begin{aligned} x_{-N} \vee y_{-N} &= (x_1, \ldots, x_M, y_{M+1}, \ldots, y_{N-1}), \\ x_{-N} \wedge y_{-N} &= (y_1, \ldots, y_M, x_{M+1}, \ldots, x_{N-1}). \end{aligned}$$

Moreover,

$$x_{-N} - x_{-N} \wedge y_{-N} = x_{-N} \vee y_{-N} - y_{-N} = (x_1 - y_1, \ldots, x_M - y_M, 0, \ldots, 0).$$

Denote this by v. For each $m \leq M$, write $v^m = (x_1 - y_1, \ldots, x_m - y_m, 0, \ldots, 0)$. Then $v^M = v$, $v^0 = 0$, and $v^m - v^{m-1} = (0, \ldots, 0, x_m - y_m, 0, \ldots, 0)$. Write $h = \ln f$. Then, for every $m \leq M$

$$\begin{aligned} &h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) \\ = &\int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, x_{M+1} \ldots, x_{N-1}, z)\,\mathrm{d}r, \\ &h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w) \\ = &\int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, y_{M+1} \ldots, y_{N-1}, w)\,\mathrm{d}r. \end{aligned}$$

Since $\partial h / \partial x_m$ is nondecreasing in x_n with $n = M+1, \ldots, N-1$ and strictly increasing in x_N,

$$\begin{aligned} &\frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, x_{M+1} \ldots, x_{N-1}, z) \\ < &\frac{\partial h}{\partial x_m}(x_1, \ldots, x_{m-1}, r, y_{m+1}, \ldots, y_M, y_{M+1} \ldots, y_{N-1}, w) \end{aligned}$$

for every r. Thus,

$$h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) < h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w).$$

Since $x_{-N} \wedge y_{-N} + v^M = x_{-N}$ and $y_{-N} + v^M = x_{-N} \vee y_{-N}$, by taking the summation of each side over $m \leq M$, we obtain

$$h(x_{-N}, z) - h(x_{-N} \wedge y_{-N}, z) < h(x_{-N} \vee y_{-N}, w) - h(y_{-N}, w).$$

That is, $A(z, w) < D(z, w)$. By swapping the roles of x_{-N} and y_{-N} (while maintaining that $z < w$), we can show that $B(z, w) < D(z, w)$.

Since $A(z, w)B(z, w) \leq C(z, w)D(z, w)$, $A(z, w) < D(z, w)$, $B(z, w) < D(z, w)$, and

$$\begin{aligned}&(C(z,w) + D(z,w)) - (A(z,w) + B(z,w)) \\ =&\frac{1}{D(z,w)}\left((C(z,w)D(z,w) - A(z,w)B(z,w)) + (D(z,w) - A(z,w))(D(z,w) - B(z,w))\right),\end{aligned}$$

we have $A(z, w) + B(z, w) < C(z, w) + D(z, w)$. Since the second term of the right-hand side of (40) is nothing but the integral of $A(z, w) + B(z, w)$ on $\{(z, w) \in \mathbb{R} \times \mathbb{R} \mid z < w\}$ and that of (41) is nothing but the integral of $C(z, w) + D(z, w)$ on the same domain, this completes the proof. □

This proposition can be extended to the case in which the domain of the function is $X_1 \times X_2 \times \cdots \times X_N$, where X_n is an interval in $\mathbb{R}$ for every n.

G Comparing kernels

Proposition 11. *For each $n = 1, 2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be differentiable and suppose that $\pi_n' < 0$. Suppose, moreover, that $\varepsilon(s; \pi_1)$ is independent of s, $\varepsilon(s; \pi_2)$ is strictly decreasing in s, and the value of the former is contained in the range of the latter. Suppose, furthermore, that there is a non-degenerate probability P on $\mathbb{R}_{++}$ s.th. $\int \pi_1(x)P(dx) = \int \pi_2(x)P(dx)$. Then, there are x_* and x^* in $\mathbb{R}_{++}$ with $x_* < x^*$ s.th. $\pi_1(x) < \pi_2(x)$ if $x < x_*$ or $x > x^*$; $\pi_1(x) > \pi_2(x)$ if $x_* < x < x^*$; and $\pi_1(x) = \pi_2(x)$ if $x = x_*$ or $x = x^*$.*

Proof of Proposition 11 Define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = \ln \pi_2(\exp z) - \ln \pi_1(\exp z)$. Then,

$$g'(z) = \frac{\pi_2'(\exp z) \exp z}{\pi_2(\exp z)} - \frac{\pi_1'(\exp z) \exp z}{\pi_1(\exp z)}.$$

Thus, g' is strictly increasing, and there are $\underline{z}$ and $\overline{z}$ s.th. $g'(\underline{z}) < 0 < g'(\overline{z})$. Then, $g'(z) \leq g'(\underline{z})$ for every $z \leq \underline{z}$ and $g'(z) \geq g'(\overline{z})$ for every $z \geq \overline{z}$. By applying the mean-value theorem to g on the interval $[z, \underline{z}]$ and the strict increasingness of g', we obtain $g(\underline{z}) \leq g'(\underline{z})(\underline{z} - z) + g(z)$, that is, $g(z) \geq -g'(\underline{z})(\underline{z} - z) + g(\underline{z})$ for every $z < \underline{z}$. As $z \to -\infty$, the right-hand side diverges to ∞. Similarly, $g(z) \geq g'(\overline{z})(z - \overline{z}) + g(\overline{z})$ for every $z > \overline{z}$. As $z \to \infty$, the right-hand side diverges to ∞. Thus, g attains its minimum (over the entire $\mathbb{R}$). Denote by $\hat{z}$ a point at which the minimum is attained. Then, $g'(\hat{z}) = 0$ by the first-order condition. Since g' is strictly increasing, $g'(z) < 0$ for every $z < \hat{z}$, and $g'(z) > 0$ for every $z > \hat{z}$. Thus, g is strictly decreasing on $(-\infty, \hat{z})$ and strictly increasing on $(\hat{z}, -\infty)$.

If $g(\hat{z}) \geq 0$, then $g(z) \geq 0$ for every z, with a strict inequality possibly except at $z = \hat{z}$. Thus, $\pi_2(x) \geq \pi_1(x)$ for every x, with a strict inequality possibly except for $x = \exp \hat{z}$, and the integral assumption is violated. Thus, $g(\hat{z}) < 0$. By the intermediate value theorem, there is a unique $z_* < \hat{z}$ s.th. $g(z_*) = 0$; and there is a unique $z^* > \hat{z}$ s.th. $g(z^*) = 0$. Let $x_* = \exp z_*$ and $x^* = \exp z^*$, to complete the proof. □

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