Sharing Model Uncertainty

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Abstract

We study efficient allocations when consumers have heterogeneous *smooth ambiguity* preferences, face model uncertainty, and consider a common set of *identifiable* models. Aggregate endowment is *ambiguous*. We characterize economies where the representative consumer is of the smooth ambiguity type and find efficient sharing rules. With heterogeneous ambiguity aversion, sharing rules exhibit systematic departures from those in *vNM-economies* and the representative consumer's nature departs from the typical single-consumer assumption, making for more compelling asset-pricing predictions. We focus on the case where models are *point-identified* but show that the insights extend when models are only *partially-identified*.

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Uncertainty, as opposed to risk, is, as ever, a major concern. Be it financial markets during the 2007-2009 crisis, policy makers confronting an emerging virus, or farmers hit by climate change - in all situations, decision makers face uncertainties that cannot be easily quantified probabilistically. It is therefore of crucial importance to understand whether and how economic institutions can deal with and possibly hedge against this uncertainty.

To analyze this issue we structure the uncertainty as *model* uncertainty. A model –comprising of parameters and distinctive mechanisms– implies a specific probabilistic forecast about the states of the world. Furthermore, we assume, in common with much of empirical economics, that the parameters and mechanisms may be estimated and *identified* from objective data. The framework we employ, of model uncertainty with identifiable models, was incorporated into decision-making under ambiguity by (Denti & Pomatto 2022). In this paper, we study the question of efficient allocations in a framework of identifiable environments, where consumers have *smooth ambiguity* preferences (Klibanoff, Marinacci & Mukerji 2005).[1]

As an example, think of co-existing stochastic models of global warming that rest on different values of parameters and different mechanisms. With the current data, models cannot be distinguished and, to use the IPCC's terminology, various *scenarios* are considered to be plausible. At the point of decision-making, the data relevant for identification are still missing. As data accumulate, scientists will be able to tell them apart: mechanisms will be identified and parameters observed.

Another example is a stochastic environment commonly applied in macrofinance, which specifies the data generating process in the macroeconomy as a regime-switching process between contractions and expansions.[2] For each

[1](Polemarchakis, Selden & Song 2022) show sufficient conditions under which one can recover individual risk and ambiguity aversion, and beliefs from asset demand functions generated by smooth ambiguity preferences.

[2]In the context of ambiguity, see (Ju & Miao 2012).

regime, the growth distribution is well-estimated. Consumers are unsure which regime governs the data in the forthcoming period. Here, we think of uncertainty about the regime as model uncertainty. Experts (e.g., NBER) decide whether the economy was in a recession, based on observations of variables from different sectors of the economy. The experts' announcement itself is an event that identifies the regime. We will use this as a running example to illustrate the concepts we introduce in the analysis and the results we obtain.

In this paper, we investigate consequences of model uncertainty on efficient allocations in an exchange economy. Consumers are unsure what would be the appropriate probability measure to apply to evaluate consumption contingent on a state space Ω and keep in consideration a set $\mathcal{P}$ of alternative probabilistic laws. Importantly, when models are identified, the usual assumption that consumption plans are contingent on events in the state space now means that they can be made effectively contingent on models too. Key to our analysis is that events in the state space not only determine endowments but also inform about the model, the probability law in play. If some consumers were ambiguity averse, efficient allocation at a state must take into account what model it informs about and not just the aggregate consumption it implies. This, as we show, can fundamentally alter the nature of efficient allocations and thereby the nature of the *representative consumer*. We study the case where consumers in the economy are heterogeneously smooth ambiguity averse and characterize and primarily focus on those economies that admit a representative consumer who is also of the smooth ambiguity type. In these economies, we derive valuable and precise insights into efficient sharing rules and the characteristics of the representative consumer. The insights obtained, initially assuming that $\mathcal{P}$ is *point-identified*,[3] robustly extend to the

[3]The true law can be recovered empirically from events in Ω.

case where models are only *set-identified*.[4]

When aggregate endowment is unambiguous[5] we show, quite generally, that efficient allocations are comonotone just as in vNM-economies. We show that an economy with a smooth ambiguity-averse representative consumer is characterized by consumers who exhibit linear risk tolerance with the same marginal risk tolerance. In such an economy, allowing for *ambiguous* aggregate endowment, we show that efficient sharing rules systematically deviate from the linearity that would arise under expected utility: more ambiguity averse consumers are allocated a larger share of the aggregate output contingent on models that are ranked worse (by the representative consumer), hence allowing them a smoother expected utility across models.

The literature on uncertainty-sharing with ambiguity aversion has not considered aggregate ambiguity. Since our analysis does allow for this, we are able to provide a foundation for macro-finance models that study effects of ambiguity aversion. The sharing rule we derive implies that the representative consumer does not have constant relative ambiguity aversion, as is widely assumed in this macro-finance literature. For instance, even if individual consumers have *constant* relative ambiguity aversion, as long as it is heterogeneous, the representative consumer will have *decreasing* relative ambiguity aversion. If expansions, compared to contractions, are associated with better distributions of aggregate output (say in the sense of FOSD), then the relative ambiguity aversion of the representative consumer is counter-cyclical.

Such a representative consumer has the potential for more compelling asset-pricing predictions than ones based on homogeneous ambiguity aversion. We give a couple of illustrations of this potential in the final section. For instance, in a Gaussian environment where the model uncertainty is simply a parameter uncertainty about the mean growth rate of the economy, we show

[4] The recovery is only up to a set of probability laws.

[5] That is, all models in $\mathcal{P}$ induce the same probability distribution over endowments.

that the pricing kernel with heterogeneous ambiguity aversion is qualitatively different than under homogeneous ambiguity aversion. The qualitative difference is significant in that it implies that the Sharpe ratio (market price of risk) is counter-cyclical. As (Lettau & Ludvigson 2010) point out this volatility and counter-cyclicality are prominent in the data and constitute a puzzle that remains to be explained.

Related literature. Efficient risk-sharing in expected-utility economies was first studied by (Borch 1962), further refined for the HARA class of utility functions by (Wilson 1968), (Cass & Stiglitz 1970) and (Hara, Huang & Kuzmics 2007) among others. (Chateauneuf, Dana & Tallon 2000) extended the comonotonicity result obtained under expected utility to Choquet expected utility with common capacity. (Billot et al. 2000), (Rigotti, Shannon & Strzalecki 2008) and (Ghirardato & Siniscalchi 2018) further studied the case in which aggregate endowment is non-risky and preferences are more general than Choquet-expected-utility preferences (including, for the two latter references, the smooth ambiguity model). (Strzalecki & Werner 2011) and (De Castro & Chateauneuf 2011) characterized properties of efficient risk-sharing when the aggregate endowment is risky but not ambiguous. (Beißner & Werner 2023) extends some of these results to cases where agents have possibly heterogeneous, non-convex ambiguity sensitive preferences. Assuming Maxmin-Expected-Utility (MEU) decision makers à la (Gilboa & Schmeidler 1989), (Wakai 2007) proves that, under HARA with common risk tolerance, efficient allocations are comonotonic. To the best of our knowledge, no paper has studied risk-sharing with *ambiguous* aggregate endowments and *heterogeneous* ambiguity aversion.

Section 1 introduces the setting and provides results on efficiency that generally apply to economies where consumers are of the smooth ambiguity type. Section 2 specializes the analysis to the case where the representative consumer is also of the smooth ambiguity type, and characterizes efficient allocations and the representative consumer. We show the results extend to the case of set-identifiable models. Section 3 illustrates the asset-pricing implications of our characterization of the representative consumer. Proofs are gathered in an Appendix. An Online-Appendix collects extensions and further supporting arguments.

1 Setting and preliminary results

1.1 Uncertainty and identifiable models

We consider a pure exchange economy under uncertainty with finite state space Ω. Let $\Delta(\Omega)$ be the set of probability distributions on Ω and let $\mathcal{P} \subset \Delta(\Omega)$. We assume that $\mathcal{P}$ is (point)-*identifiable*, i.e., there exists a *kernel* function $k : \Omega \to \mathcal{P}$ such that for all $\mathbf{P} \in \mathcal{P}$, $\mathbf{P}(\{\omega : k(\omega) = \mathbf{P}\}) = 1$. Note, given that Ω is finite, identifiability has the consequence that $\mathcal{P}$ has to be finite as well. Each $\mathbf{P} \in \mathcal{P}$ is a possible data generating process, or *probabilistic model* (or simply *model*, for short), governing the state. Identifiability makes elements of $\mathcal{P}$ objective descriptions, inferable from data, that is, events in Ω.[6]

An important feature of *identified* models is that two distinct measures $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ are singular to each other because $\mathbf{P}$ puts mass 1 on the event $\{\omega | k(\omega) = \mathbf{P}\} \equiv \Omega_{\mathbf{P}}$ and $\mathbf{Q}$ puts mass 1 on the event $\{\omega | k(\omega) = \mathbf{Q}\} \equiv \Omega_{\mathbf{Q}}$. Thus, $(\Omega_{\mathbf{P}})_{\mathbf{P} \in \mathcal{P}}$ constitutes a partition of Ω. We assume throughout that for all $\mathbf{P} \in \mathcal{P}$ and all $\omega \in \Omega_{\mathbf{P}}$, $\mathbf{P}(\omega) > 0$, i.e., $\operatorname{supp}(\mathbf{P}) = \Omega_{\mathbf{P}}$.

Example 1 (Ellsberg)**.** *A ball is drawn from an urn that contains red, blue and yellow balls. The composition of the urn is unknown, but is verifiable ex post. A state of the world* $\omega = (c, \gamma)$ *specifies the color of the ball drawn,*

[6]In Section 2.3, we extend our analysis to the case where $\mathcal{P}$ is only *partially* or *set-identifiable*, in which case the kernel function is set-valued and associates to each ω a subset of $\mathcal{P}$.

$c \in \{r, b, y\}$, and the composition of the urn $\gamma \in \Delta(\{r, b, y\})$. The set of probabilistic models $\mathcal{P} = \{\mathbf{P}_\gamma\}$ is indexed by the composition γ and each $\mathbf{P}_\gamma$ assigns probability 1 to the event $\{r, b, y\} \times \{\gamma\}$. The identifying kernel is given by $k((c, \gamma)) = \mathbf{P}_\gamma$.

The Ellsberg example has a very specific feature: the state space can be expressed as the product of two components, one of which completely determines the probabilistic model in play. This is not the case in general, as the following example shows. The example will be used to illustrate our constructs and motivate the asset pricing exercise in Section 3.

Running Example. *Consider an economy that may be in one of two regimes, Boom (B) or bust (b), in a given period. Estimations based on historical data associates a regime with a particular probability distribution on (aggregate) endowment. Over the course of a period, the endowment realized, together with a variety of observations on the credit market, labor market, etc., which enable the NBER expert committee to determine and publicly announce the regime in operation in the period and thus, effectively, the probability distribution.*

To represent this, let Ω have three components that relate to the state of the economy, each of them taking a high or low value: financial state $(\underline{m}, \bar{m})$, state of the labour market $(\underline{\ell}, \bar{\ell})$, current endowment $(\underline{x}, \bar{x})$. These variables are observable. We therefore have $\Omega = \{\underline{m}, \bar{m}\} \times \{\underline{\ell}, \bar{\ell}\} \times \{\underline{x}, \bar{x}\}$. $\mathcal{P}$ has two elements $\mathbf{P}_B \in \Delta(\Omega)$ and $\mathbf{P}_b \in \Delta(\Omega)$. The kernel $k : \Omega \to \mathcal{P}$ is defined according to the way the NBER assesses the state of the economy: say that the NBER calls a bust when at least two variables are low, while any state with at least two high variables identifies the Boom regime. Hence, $k(\underline{m}, \bar{\ell}, \underline{x}) = \mathbf{P}_b$, etc. So, $\Omega_b = \{(\underline{m}, \underline{\ell}, \underline{x}), (\underline{m}, \underline{\ell}, \bar{x}), (\underline{m}, \bar{\ell}, \underline{x}), (\bar{m}, \underline{\ell}, \underline{x})\}$, and $\Omega_B = \Omega \setminus \Omega_b$. Notice $\mathbf{P}_B(\Omega_B) = 1$ and $\mathbf{P}_b(\Omega_b) = 1$. Note, we have assumed that $\mathbf{P}_B$ and $\mathbf{P}_b$ are given exogenously. They could be endogenized by using *the history of the economy to estimate the distributions.*[7]

1.2 A pure exchange economy

Our pure exchange economy consists of one good, and finitely many consumers, $i = 1, \ldots, I$. Endowments and consumption may be uncertain, contingent on Ω. (Aggregate) endowment is given by a function $\bar{X} : \Omega \to \mathbb{R}_+$. Let $\bar{X}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{R}_+$ be the restriction of $\bar{X}$ to $\Omega_{\mathbf{P}}$ and write $\bar{X} = (\bar{X}^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}$. Similarly, a contingent consumption plan for consumer i is a mapping $X_i : \Omega \to \mathbb{R}_+$. We denote by $X_i^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{R}_+$ the restriction of X_i to $\Omega_{\mathbf{P}}$ and write $X_i = (X_i^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}}$.

Let $f : \Omega \to \mathbb{R}$, and say that the *range* of f is *model-independent* if $f(\Omega_{\mathbf{P}}) = f(\Omega_{\mathbf{Q}})$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$. Hence, the range of aggregate endowment is model-independent if the set of possible realizations is the same under various models. For some of the propositions to come, model-independence of the range of aggregate endowment will be (explicitly) invoked. We say that f is *unambiguous* if $\mathbf{P} \circ f^{-1} = \mathbf{Q} \circ f^{-1}$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$. So, the aggregate endowment $\bar{X}$ is unambiguous if its distribution is independent of $\mathbf{P}$, that is, $\mathbf{P} \circ (\bar{X}^{\mathbf{P}})^{-1} = \mathbf{Q} \circ (\bar{X}^{\mathbf{Q}})^{-1}$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$. Note, $\bar{X}$ is unambiguous implies that its range is model-independent.

Running Example continued. *Assume that $\mathbf{P}_B$ assigns probability 1/4 to each state in Ω_B and $\mathbf{P}_b$ assigns probability 1/4 to each state in Ω_b. The distribution $\mathbf{P}_B$ (resp. $\mathbf{P}_b$) induces a distribution, P_B (resp. P_b) on $\{\underline{x}, \bar{x}\}$.*

[7] Assume that the probability measure over the product state space Ω_b^∞ is i.i.d. and, analogously, that the one over Ω_B^∞ is also i.i.d., with $\mathbf{P}_b$ (and $\mathbf{P}_B$) being the marginal on the single coordinate Ω_b (respectively, Ω_B). Think of a single coordinate as a snapshot at a point in time. Then the empirical frequency conditional on b, and that conditional on B, identify $\mathbf{P}_b$ and $\mathbf{P}_B$, respectively. Suppose further that our example is set at a point in history when a long enough sample has been observed so that the estimates are seen to be stable and, consequently, accepted as firm. ((Klibanoff, Mukerji & Seo 2014), (Klibanoff et al. 2022), and Example 1 in (Denti & Pomatto 2022) relate such environments to ambiguity.)

Given the equiprobability assumption, we have: $P_B(\bar{x}) = 3/4$ *and* $P_B(\underline{x}) = 1/4$*, while* $P_b(\bar{x}) = 1/4$ *and* $P_b(\underline{x}) = 3/4$*. Equality of the supports of* P_b *and* P_B *implies that the range of endowment is model-independent. Still, the endowment is ambiguous since* $P_b \neq P_B$*, that is, each possible realization has a different probability of occurrence.*[8]

Let $u_i : \mathbb{X}_i \to \mathbb{R}$ be a Bernoulli utility function, twice continuously differentiable, strictly increasing and strictly concave on its domain $\mathbb{X}_i$, an interval in $\mathbb{R}$. Let $\phi_i : u_i(\mathbb{X}_i) \to \mathbb{R}$ be twice continuously differentiable, strictly increasing, and concave. Let $E^{\mathbf{P}}$ be the expectation operator on random variables defined on Ω under the probability measure $\mathbf{P}$. Note, $E^{\mathbf{P}} u_i(X_i) = E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)$. Consumer i's preferences are represented by the *identifiable smooth ambiguity* utility function:[9]

$$U_i(X_i) = \int_{\mathcal{P}} \phi_i\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right) \mu(d\mathbf{P}) \tag{1}$$

Ambiguity attitudes, possibly heterogeneous, are captured by the properties of ϕ_i: consumer i is *(strictly) ambiguity averse* if ϕ_i is (strictly) concave and *ambiguity neutral* if ϕ_i is linear, in which case the consumer is of the expected utility type. Finally, $\mu \in \Delta(\Delta(\Omega))$ is a full support prior over $\mathcal{P}$, assumed to be common across consumers.

Remark 1. The fact that μ is common ensures that the risk-sharing analysis is not driven by differences in beliefs (i.e., speculation) but rather by differences in risk and ambiguity attitudes across consumers (i.e., insurance

[8] Given that the NBER announces the regime publicly, one could use a reduced form, where the state space is simply $\{0,1\} \times \{\underline{x}, \bar{x}\}$ with $k(0,x) = \mathbf{P}_b$, $k(1,x) = \mathbf{P}_B$, for $x = \underline{x}, \bar{x}$. This is a coarsening of the original state space. Actually, this is a general property, given identifiability: when the range of endowment is model-independent, $\bar{X}(\Omega) \times \mathcal{P}$ is a coarsening of Ω. We shall employ such a product state space in our asset pricing exercise in Section 3.

[9] The identifiable smooth representation was introduced and axiomatized by (Cerreia-Vioglio et al. 2013) taking $\mathcal{P}$ as a primitive. (Denti & Pomatto 2022) provides an axiomatic foundation where $\mathcal{P}$ is revealed by choice behavior.

reasons). Taken together with heterogeneity in ϕ_i, it is similar to the condition that MEU maximizers have *distinct* sets of priors with a non-empty intersection,[10] as the following probability matching exercise shows. Consider a bet on model $\mathbf{P}$, paying c^* on event $\Omega_{\mathbf{P}}$ and c_* off it, and a lottery ℓ^π which pays c^* with probability π and c_* with probability $1-\pi$, where $c^* > c_*$. For ϕ strictly concave, there is an interval $[\underline{\pi}, \bar{\pi}] \ni \mu(\mathbf{P})$ such that for every $\pi \in [\underline{\pi}, \bar{\pi}]$, the bet on $\Omega_{\mathbf{P}}$ is less desirable than ℓ^π and the bet on the complementary event $\Omega \setminus \Omega_{\mathbf{P}}$ is *also* less desirable than $\ell^{1-\pi}$. Furthermore, the interval is wider, the more ambiguity averse the consumer.[11] Hence, the consumers act *as if* their beliefs that the model $\mathbf{P}$ is true are described by a probability interval containing $\mu(\mathbf{P})$. Given heterogeneity in ambiguity aversion and commonality of μ, these intervals are different across consumers but have a non-empty intersection that includes $\mu(\mathbf{P})$. Hence, there is shared information about the likelihood of models but this information is acted upon with differing degrees of trust by heterogeneously ambiguity averse consumers.

Running Example continued. *The economy here is a snapshot, at a given time, of the dynamic workhorse model in macro-finance presented in, for example, (Cecchetti, Lam & Mark 1990) and (Kandel & Stambaugh 1991), where at each period there are two possible distributions,* $\mathbf{P}_b$ *and* $\mathbf{P}_B$*, that could be at play. In (Cecchetti, Lam & Mark 1990), consumers observe the regime and the Markovian transition probabilities are the consumers' conditional beliefs about the regime in the next period, which we may take to coincide with* $(\mu, 1-\mu)$*, where* $\mu = \mu(\mathbf{P}_B)$*. Hence,* (1) *reduces to* $U_i(X_i) = \mu\phi_i\left(E^{\mathbf{P}_B}u_i\left(X_i^{\mathbf{P}_B}\right)\right) + (1-\mu)\phi_i\left(E^{\mathbf{P}_b}u_i\left(X_i^{\mathbf{P}_b}\right)\right)$*. In the traditional literature the possibility of two regimes, captured by* $\mathcal{P} = \{\mathbf{P}_b, \mathbf{P}_B\}$*, is treated in*

[10]In (Billot et al. 2000) and (Rigotti, Shannon & Strzalecki 2008) it is the condition that ensures that efficiency entails full insurance.

[11]see Online-Appendix A for calculations.

an ambiguity neutral fashion whereas our analysis allows for ambiguity aversion. Furthermore, contracts effectively contingent on events identifying each regime are available.

1.3 Efficient allocations

An allocation $(X_i)_{i=1,\ldots,I}$ is *feasible* if $X_i(\omega) \in \mathbb{X}_i$ for all i, all $\omega \in \Omega$ and $\sum_{i=1}^{I} X_i \leq \bar{X}$.

Definition 1. *Let $(X_i)_{i=1,\ldots,I}$ be a feasible allocation. Say that $(X_i)_{i=1,\ldots,I}$ is*

1. efficient *if there is no feasible allocation $(Y_i)_{i=1,\ldots,I}$ on Ω with $U_i(Y_i) \geq U_i(X_i)$ for all i, with at least one strict inequality;*
2. $\mathbf{P}$-conditionally efficient *for $\mathbf{P} \in \mathcal{P}$, if there is no feasible allocation $\left(Y_i^{\mathbf{P}}\right)_{i=1,\ldots,I}$ with $E^{\mathbf{P}}\left(u_i\left(Y_i^{\mathbf{P}}\right)\right) \geq E^{\mathbf{P}}\left(u_i\left(X_i^{\mathbf{P}}\right)\right)$ for all i, with at least one strict inequality;*
3. conditionally efficient *if $(X_i^{\mathbf{P}})_{i=1,\ldots,I}$ is $\mathbf{P}$-conditionally efficient for all $\mathbf{P}$.*

For convex preferences, efficient allocations can be found by solving the utilitarian welfare (Negishi) problem of maximizing the weighted sum of utilities $\sum_i \lambda_i U_i(X_i)$ over all feasible allocations $(X_i)_{i=1,\ldots,I}$ for individual weights $\lambda_i \geq 0$. The resulting value function V defines the preferences of a *representative consumer*.[12] In our identifiable context, Negishi's problem may be written as:

$$\begin{aligned} V(\bar{X}) = \max_{(X_i)_{i=1,\ldots,I}} \quad & \sum_i \lambda_i U_i(X_i) \\ \text{subject to} \quad & \sum_i X_i^{\mathbf{P}} \leq \bar{X}^{\mathbf{P}} \quad \text{for all } \mathbf{P} \in \mathcal{P}. \end{aligned} \tag{2}$$

[12] This notion of a representative consumer is common in the context of asset pricing; see e.g. Chapter 1, eqn (6) of (Duffie 2001) and should not be confused with the more demanding notion of aggregation of (Gorman 1959)).

The fact that we can determine the optimal allocation *model by model* in an identifiable context has the important consequence that the prior μ does not play a role in determining efficient allocations. In fact, we have

$$V\left(\bar{X}\right) = \sum_{\mathbf{P}\in\mathcal{P}} \mu(\mathbf{P}) V^{\mathbf{P}}\left(\bar{X}^{\mathbf{P}}\right), \tag{3}$$

where

$$\begin{array}{lcl} V^{\mathbf{P}}(\bar{X}^{\mathbf{P}}) \equiv \max\limits_{\left(X_i^{\mathbf{P}}\right)_{i=1,\dots,I}} & & \sum\limits_i \lambda_i \phi_i\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right) \\ \text{subject to} & & \sum\limits_i X_i^{\mathbf{P}} \leq \bar{X}^{\mathbf{P}}. \end{array} \tag{4}$$

We can thus fix a probabilistic model $\mathbf{P} \in \mathcal{P}$ and maximize the weighted sum $\sum_i \lambda_i \phi_i\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right)$ over $\mathbf{P}$-feasible allocations, i.e., allocations satisfying the feasibility constraint on $\Omega_{\mathbf{P}}$. The ambiguity attitudes ϕ_i are strictly increasing so the solutions to (4) coincide with efficient *risk sharing* under model $\mathbf{P}$, i.e., the efficient risk sharing in an economy populated by vNM-expected utility consumers, which we will refer to as a vNM-economy.

We have thus shown the following Proposition:

Proposition 1. *Every efficient allocation is conditionally efficient.*

Let $c_i^{\mathbf{P}}(X_i^{\mathbf{P}}) = u_i^{-1}\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right)$ be the certainty equivalent of consumer i's consumption plan under model $\mathbf{P}$, and let $v_i : \mathbb{X}_i \to \mathbb{R}$ be defined by $v_i = \phi_i \circ u_i$. The smooth ambiguity utility can be written in two equivalent ways: as in (1), or as an expected utility of certainty equivalents:

$$U_i\left(X_i\right) = \int_{\mathcal{P}} v_i\left(c_i^{\mathbf{P}}(X_i^{\mathbf{P}})\right) \mu(d\mathbf{P}). \tag{5}$$

The separability across models inherent in (3) and in (5) suggests a formulation of the representative consumer's evaluation of an aggregate endowment *via* a two-stage Negishi maximization algorithm: first determine, model by model, the conditionally-efficient allocations; convert these to certainty equivalents (using the distribution associated with the model) and then, in the second stage, maximize over these certainty equivalents across models. Formally, denoting by $PO(\bar{X}^{\mathbf{P}})$ the set of $\mathbf{P}$-conditionally efficient allocations, we can represent the representative consumer's preferences by:

$$V\left(\bar{X}\right) = \int_{\mathcal{P}} \max_{c \in \mathcal{C}^{\mathbf{P}}} \sum_i \lambda_i v_i(c_i) \mu(d\mathbf{P}) \tag{6}$$

where $\mathcal{C}^{\mathbf{P}} = \left\{c \in \mathbb{R}^I : c_i \leq c_i^{\mathbf{P}}(X_i) \text{ for some } X \in PO(\bar{X}^{\mathbf{P}})\right\}$.

However, the preferences in (6) are not necessarily of the smooth ambiguity type –see Remark 2 at the end of Section 2.1. In the next section we characterize economies in which this is the case.

Part 1 in Proposition 2 below follows from Proposition 1 and known results for vNM-economies (recall our maintained assumption that $\operatorname{supp}(\mathbf{P}) = \Omega_{\mathbf{P}}$). Part 2 says that if the aggregate endowment is unambiguous (which, recall, implies that the range of endowment is model-independent), efficient allocation provides complete insurance against model uncertainty: allocations do not depend on models but depend exclusively on the realizations of $\bar{X}$. Taking the two parts together, if the aggregate endowment is unambiguous an efficient allocation is comonotone, just as in a vNM-economy.

Proposition 2.

1. *If $(X_i)_{i=1,\dots,I}$ is an interior efficient allocation, then for any fixed $\mathbf{P} \in \mathcal{P}$, the allocation $(X_i^{\mathbf{P}})_{i=1,\dots,I}$ is $\mathbf{P}$-comonotone, that is for all $\omega, \omega' \in \Omega_{\mathbf{P}}$ and all i, $X_i^{\mathbf{P}}(\omega) \leq X_i^{\mathbf{P}}(\omega')$ if and only if $\bar{X}^{\mathbf{P}}(\omega) \leq \bar{X}^{\mathbf{P}}(\omega')$.*

2. *Assume ϕ_i strictly concave for all i and the aggregate endowment is unambiguous. Then, the allocation $Y = (Y^{\mathbf{P}})_{\mathbf{P} \in \mathcal{P}}$ is efficient if and only if Y is conditionally efficient and $Y_i^{\mathbf{P}}((\bar{X}^{\mathbf{P}})^{-1}(x)) = Y_i^{\mathbf{Q}}((\bar{X}^{\mathbf{Q}})^{-1}(x))$ for all i, all $x \in \bar{X}(\Omega)$ and all $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$.*

If aggregate endowment is ambiguous, efficient allocations are not necessarily comonotone.[13] However, under ambiguity aversion, since consumers care about smoothing welfare across models it seems natural that efficiency would require that consumers' welfare move in the same direction across models, a property we call *Expected-Utility-comonotonicity.*

Definition 2. *An allocation* $(X_i)_{i=1,\dots,I}$ *is* Expected-Utility-comonotone *(or EU-comonotone) if for every* i,j *and* $\mathbf{P},\mathbf{Q} \in \mathcal{P}$*,* $E^{\mathbf{P}} u_i(X_i^{\mathbf{P}}) \leq E^{\mathbf{Q}} u_i(X_i^{\mathbf{Q}})$ *if and only if* $E^{\mathbf{P}} u_j(X_j^{\mathbf{P}}) \leq E^{\mathbf{Q}} u_j(X_j^{\mathbf{Q}})$*.*

The following proposition provides a sufficient condition for efficient allocation to be EU-comonotone: that the distributions on aggregate endowment induced by models are ordered by FOSD.

Proposition 3. *Assume the range of* $\bar{X}$ *is model-independent. Let* $(X_i)_{i=1,\dots,I}$ *be an efficient allocation. Suppose that, for* $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$*, the distribution of the aggregate endowment* $\bar{X}$ *under* $\mathbf{P}$*,* $\mathbf{P} \circ \bar{X}^{-1}$*, is first-order stochastically dominated by* $\mathbf{Q} \circ \bar{X}^{-1}$*. Then* $E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right) \leq E^{\mathbf{Q}} u_i\left(X_i^{\mathbf{Q}}\right)$ *for every* i*.*

If the set $\{\mathbf{P} \circ \bar{X}^{-1} \mid \mathbf{P} \in \mathcal{P}\}$ is totally ordered by FOSD, then EU-comonotonicity is obtained over the entire set $\mathcal{P}$; in other words, all consumers rank models the same way. Actually, under the condition that the distributions on aggregate endowment induced by models are ordered by FOSD, EU-comonotonicity, together with conditional efficiency, exhaust the properties of efficient allocations. This can be made formal as follows: under some technical conditions, if an allocation is conditionally efficient and satisfies EU-comonotonicity given a profile $(u_i)_{i=1,\dots,I}$, then there exists a profile of concave and twice-differentiable $(\phi_i)_{i=1,\dots,I}$ such that the allocation is efficient.[14] In the next section we show how EU-comonotonicity is obtained

[13] This follows from Proposition 6 under heterogeneity of ambiguity aversion. See the discussion in the running example after the Proposition.

[14] Proposition 10 in (Hara et al. 2022).

without restricting the class of models, but, instead, by restricting risk attitudes.

2 Representative consumers with smooth ambiguity preferences

We identify economies that admit a *smooth ambiguity representative consumer*. In such economies, we characterize efficient sharing rules and show how risk and ambiguity attitudes of the representative consumer relate to those of the individual consumers. An advantage of studying these economies is that the insights obtained are robust in the sense that they extend to the case where models are only *set-identified*, as we show in Section 2.3.

A consumer i's utility function u_i satisfies linear risk tolerance with parameters (b_i, a_i) if

$$-\frac{u_i'(x_i)}{u_i''(x_i)} = a_i + b_i x_i \tag{7}$$

holds on the domain $a_i + b_i x_i > 0$. This is the well-known class of HARA utility functions. For $b_i = 0$, the function is CARA with index $\frac{1}{a_i}$. Quadratic functions correspond to $b_i = -1$. When $b_i > 0$ and $a_i = 0$ the function exhibits CRRA with index $\frac{1}{b_i}$. When $b_i > 0$ and $a_i \neq 0$, the class of functions is of the "shifted power" type.[15] The HARA class thus covers the gamut of Bernoulli utility functions considered in economics and finance.

Proposition 16.13 in (Magill & Quinzii 1996), based on (Wilson 1968) and (Cass & Stiglitz 1970), asserts that efficient allocations in a vNM-economy satisfy a linear sharing rule if and only if consumers' utility functions exhibit linear risk tolerance with common marginal risk tolerance (see also (Hara, Huang & Kuzmics 2007)).

[15] (Back 2017) Section 1.3.

Condition 1. *The functions $(u_i)_{i=1,\dots,I}$ are HARA with common marginal risk tolerance (henceforth CMRT), that is, $b_i = b$ for all i.*

In what follows we show that, in general, an economy populated by smooth ambiguity consumers will have a smooth ambiguity representative consumer if and only if the economy satisfies Condition 1.

2.1 A nested Negishi approach

In a vNM-economy we may find the efficient allocations by solving the following program for weights λ_i where $\bar{x}$ is a realized aggregate consumption level:

$$\begin{aligned} u(\bar{x}) \equiv \max_{(x_i)_{i=1,\dots,I}} & \sum_{i=1,\dots,I} \lambda_i u_i(x_i) \\ \text{subject to} & \sum_{i=1,\dots,I} x_i \le \bar{x}. \end{aligned} \tag{8}$$

As is well-known,[16] under Condition 1, the value function u does not depend on the λ_i and the representative consumer in the vNM-economy has expected utility preferences, with Bernoulli utility function u. Let $c_u^{\mathbf{P}}(\bar{X}) = u^{-1}\left(E^{\mathbf{P}} u(\bar{X})\right)$.[17] Lemma 1 notes a key property of these economies.

Lemma 1. *Let $(X_i^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P},i=1,\dots,I}$ be conditionally efficient. Then, under Condition 1, $\sum_{i=1,\dots,I} c_i^{\mathbf{P}}(X_i^{\mathbf{P}}) = c_u^{\mathbf{P}}(\bar{X})$ for all $\mathbf{P}$.*

Lemma 1 delivers additivity of the certainty equivalents at conditionally efficient allocations. This (and Proposition 1) allows us to characterize an efficient allocation in two steps. First, solve program (4) for **P**-conditionally ef-

[16](LeRoy & Werner 2014), Section 16.8. or (Gollier 2001), section 21.4.1.

[17]The characterization of the representative consumer's utility u is in (Wilson 1968) and (Cass & Stiglitz 1970). It is in the same HARA subclass as the individual consumers'. For example, when u_is are CARA with parameter α_i, u is CARA u with parameter α, $\sum_i(\alpha/\alpha_i) = 1$. When u_is are shifted power with parameters (b, a_i), u is shifted power with parameters $(b, \sum_i a_i)$.

ficient allocations (the inner program), yielding aggregate $\mathbf{P}$-certainty equivalents that can be then allocated across models by solving the following Negishi (outer) programs:

$$\begin{aligned} v(c_u) \equiv \max_{(c_i)_{i=1,\ldots,I}} \quad & \sum_{i=1,\ldots,I} \lambda_i v_i\left(c_i\right) \\ \text{subject to} \quad & \sum_{i=1,\ldots,I} c_i \leq c_u. \end{aligned} \tag{9}$$

Condition 1 allows us to simplify (6) by making the constraint $\mathcal{C}^{\mathbf{P}}$ linear. The outer program implements efficient sharing of $\mathbf{P}$-contingent aggregate certainty equivalents: think of $(c_i)_{i=1,\ldots,I} = \left(c_i^{\mathbf{P}}(X_i^{\mathbf{P}})\right)_{i=1,\ldots,I}$ as an efficient allocation of the aggregate resource $c_u = c_u^{\mathbf{P}}(\bar{X})$ in "state $\mathbf{P}$" across consumers with "Bernoulli utility v_i".

Let $\phi = v \circ u^{-1}$, then $V^{\mathbf{P}}(\bar{X}) = v(c_u^{\mathbf{P}}(\bar{X})) = \phi(E^{\mathbf{P}}u(\bar{X}))$. Analogous to the solution of a vNM-economy problem, the allocation $(c_i)_{i=1,\ldots,I}$, the solution of program (9), is comonotone with respect to c_u across states $\mathbf{P}$. Hence, the efficient allocation $(X_i)_{i=1,\ldots,I}$ is EU-comonotone.[18]

Proposition 4. *Let $(u_i)_{i=1,\ldots,I}$ satisfy Condition 1. Then,*

1. *The representative consumer's utility V is of the smooth ambiguity form: $V\left(\bar{X}\right) = \int_{\mathcal{P}} \phi\left(E^{\mathbf{P}}u\left(\bar{X}\right)\right)\mu(d\mathbf{P})$ with $\phi = v \circ u^{-1}$, where u is the value function of (8) and v is the value function of (9). Moreover, $\phi'' \leq 0$, and $\phi'' = 0$ if and only if $\phi_i'' = 0$ for all i.*

2. *An efficient allocation is EU-comonotone.*

A consequence of Part 1 of Proposition 4 is that the representative consumer is strictly ambiguity averse as soon as there is one strictly ambiguity averse consumer in the economy. This is different from the case of pure risk sharing, where if some consumers were risk neutral they would bear all

[18]Neither the validity of the nested Negishi approach (9) nor Proposition 4 requires conditions on v_i (beyond concavity).

the risk, and the representative consumer would be risk neutral. Since EU-comonotonicity of an allocation is equivalent to the comonotonicity of the associated certainty equivalents, the second part of the proposition states that, strikingly, all consumers rank models in exactly the same way at an efficient allocation, *without any restrictions on the class of models.*

Condition 1 is not only sufficient for obtaining a smooth ambiguity representative consumer but is also necessary. If it is not satisfied we can find an economy with heterogeneous smooth ambiguity averse consumers (with common μ) such that the representative consumer's utility function as defined in (2) is not of the smooth ambiguity type.

Proposition 5. *Suppose the profile $(u_i)_{i=1,\dots,I}$ does not satisfy Condition 1. Then, there are μ, $(\phi_i)_{i=1,\dots,I}$ and $\bar{X}$ such that, if V is defined as in* (2)*, there is no pair u and ϕ so that $V\left(\bar{Y}\right) = \int_{\mathcal{P}} \phi\left(E^{\mathbf{P}} u\left(\bar{Y}^{\mathbf{P}}\right)\right) \mu(d\mathbf{P})$ for all $\bar{Y}$.*

Outline of proof for Proposition 5:[19] Consider a two-consumer, four-state economy, $\Omega = \{\omega_{\mathbf{P}}, \omega'_{\mathbf{P}}, \omega_{\mathbf{Q}}, \omega'_{\mathbf{Q}}\}$, with $\mathcal{P} = \{\mathbf{P}, \mathbf{Q}\}$, $\Omega_{\mathbf{P}} = \{\omega_{\mathbf{P}}, \omega'_{\mathbf{P}}\}$ and $\Omega_{\mathbf{Q}} = \{\omega_{\mathbf{Q}}, \omega'_{\mathbf{Q}}\}$. Fix u_1 and u_2 that violate CMRT, thus contradicting Condition 1, and assume $\mathbf{P}(\omega_{\mathbf{P}}) > \mathbf{Q}(\omega_{\mathbf{Q}})$. Since Condition 1 is not satisfied, we can pick a deterministic allocation at which the two consumers have different marginal risk tolerance. Then, we can construct an allocation, in a neighborhood of this deterministic allocation in the following way. Assume that : $\bar{X}(\omega_{\mathbf{P}}) = \bar{X}(\omega_{\mathbf{Q}}) > \bar{X}(\omega'_{\mathbf{P}}) = \bar{X}(\omega'_{\mathbf{Q}})$. Take two distinct $\mathbf{P}$-conditionally efficient (and hence comonotone with respect to $\bar{X}$) allocations, one associated with model $\mathbf{P}$, the other with model $\mathbf{Q}$. Pick these allocations such that $X_1^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ and $X_1^{\mathbf{P}}(\omega'_{\mathbf{P}}) > X_1^{\mathbf{Q}}(\omega'_{\mathbf{Q}})$, and hence, $X_2^{\mathbf{P}}(\omega_{\mathbf{P}}) < X_2^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ and $X_2^{\mathbf{P}}(\omega'_{\mathbf{P}}) < X_2^{\mathbf{Q}}(\omega'_{\mathbf{Q}})$. Obviously, we get that $E^{\mathbf{P}}u_1(X_1^{\mathbf{P}}) > E^{\mathbf{Q}}u_1(X_1^{\mathbf{Q}})$. How about consumer 2? Choose the $\mathbf{P}$ and $\mathbf{Q}$-conditionally efficient allocations "close enough" such that $X_2^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_2^{\mathbf{Q}}(\omega'_{\mathbf{Q}})$. Then, if $\mathbf{P}(\omega_{\mathbf{P}})$ is

[19] The full argument can be found in Online-Appendix C.

sufficiently high and $\mathbf{Q}(\omega_{\mathbf{Q}})$ sufficiently low, we obtain that $E^{\mathbf{P}}u_2(X_2^{\mathbf{P}}) > E^{\mathbf{Q}}u_2(X_2^{\mathbf{Q}})$. Hence, the overall allocation is EU-comonotone and, consequently, efficient for some (ϕ_1, ϕ_2) –see footnote 14.

(Wilson 1968) famously showed that the risk tolerance of the representative consumer in a vNM-economy at an efficient allocation is the sum of individual consumers' risk tolerances at this allocation. In this case, if 1 has higher marginal tolerance than 2, then the representative consumer's risk tolerance is "lower", as it were, under **P** than under **Q** (since contingent on **P**, consumer 1 (resp. 2) has more (resp. less) than under **Q**, irrespective of the realization of endowment). Hence, the representative consumer's Bernoulli utility is not just dependent on the aggregate endowment but also on how it is allocated. Since we cannot obtain a Bernoulli u that depends exclusively on $\bar{X}$, the overall value function V does not have the requisite smooth ambiguity form, $V(\bar{X}) = \mu(\mathbf{P})\phi(E^{\mathbf{P}}u(\bar{X})) + \mu(\mathbf{Q})\phi(E^{\mathbf{Q}}u(\bar{X}))$. □

Remark 2. *The foregoing argument illustrates why the preferences in* (6) *are not necessarily of the smooth ambiguity type.*

2.2 Sharing rules and the representative consumer's ambiguity aversion

In this section, we characterize sharing rules in economies with a smooth ambiguity representative consumer. For this purpose, given Propositions 4 and 5, we may exclusively consider economies satisfying Condition 1.

Note, v_i and u_i are both defined on the consumption space, unlike ϕ_i, which is defined on the domain of (expected) utilities. It is therefore natural to require v_i to have the same parametric form as u_i. Imposing that $(v_i)_{i=1,\ldots,I}$ is HARA, further to Condition 1 on $(u_i)_{i=1,\ldots,I}$, enables us to characterize $\phi = v \circ u^{-1}$, the ambiguity aversion of the representative consumer, as a function of the individual consumers' ambiguity aversion. We consider here the case where u_i's and v_i's have strictly positive marginal risk tolerance, which is the traditional focus in macro and finance models.

Condition 2. *There exists α s.th. for all i, there exist ζ_i and γ_i s. th.:*

(i) $\forall i = 1, \ldots, I$, $u_i : \mathbb{X}_i \to \mathbb{R}$ *is HARA with parameters* $(\frac{1}{\alpha}, -\frac{\zeta_i}{\alpha})$, $\alpha > 0$.

(ii) $\forall i = 1, \ldots, I$, $v_i : \mathbb{X}_i \to \mathbb{R}$ *is HARA with parameters* $(\frac{1}{\gamma_i}, -\frac{\zeta_i}{\gamma_i})$ *with* $\gamma_i \geq \alpha$.

The functions u_i and v_i satisfying Condition 2 are of the shifted power type, e.g., for u_i, defined on $\mathbb{X}_i = (\zeta_i, \infty)$:

$$u_i(x_i) = \begin{cases} \frac{\alpha}{1-\alpha}\left(\frac{x_i-\zeta_i}{\alpha}\right)^{1-\alpha} & \text{if} \quad \alpha \neq 1, \\ \ln(x_i - \zeta_i) & \text{otherwise,} \end{cases}$$

and so, $-\frac{u_i''(x)}{u_i'(x)} = \frac{\alpha}{x-\zeta_i}$. Hence, the relative risk aversion coefficient, relative to effective consumption $z \equiv x - \zeta_i$, is α. Define the *relative ambiguity aversion* coefficient, *relative to effective consumption*, for consumer i by[20]

$$RAA_{\phi_i}(z) \equiv -\frac{\phi_i''(u_i(z+\zeta_i))}{\phi_i'(u_i(z+\zeta_i))} u_i'(z+\zeta_i)z.$$

Under Condition 2, $RAA_{\phi_i}(z) = \gamma_i - \alpha$ is positive for all $i = 1, \ldots, I$ and independent of z. Notice that Condition 2 does *not* require the v_i's to satisfy CMRT. This is significant since it allows for heterogeneity in the consumers' relative ambiguity aversion.

Proposition 6 first characterizes the efficient sharing rule.[21] Classic results establish that it is linear contingent on a model $\mathbf{P} \in \mathcal{P}$. Across models, we show, the rule adjusts by making the slope coefficient model-contingent. Secondly, it characterizes the representative consumer's relative ambiguity aversion, denoted RAA_ϕ, where ϕ is as in Proposition 4, establishing that

[20] See Online-Appendix B.

[21] Online-Appendix E contains further material describing the efficient rule.

it is decreasing if and only if ambiguity aversion is heterogeneous in the economy.[22]

Proposition 6. *Assume the range of $\bar{X}$ is model-independent. Let $(X_i)_{i=1,\dots,I}$ be an efficient allocation. Then, under Condition 2:*

1. *there exist functions $\theta_i : (0, \infty) \to (0, 1)$ with $\sum_i \theta_i(z) = 1$, and*

$$X_i^{\mathbf{P}} = \theta_i(c_u^{\mathbf{P}}(\bar{X}) - \zeta) \cdot (\bar{X} - \zeta) + \zeta_i$$

   where $\zeta = \sum_i \zeta_i$. Furthermore, $\forall i, j$, and $\forall z > 0$,

   (a) $\frac{d}{dz}\left(\frac{\theta_j(z)}{\theta_i(z)}\right) > (=)\ 0$ *iff* $RAA_{\phi_i} = \gamma_i - \alpha > (=)\ RAA_{\phi_j} = \gamma_j - \alpha$;

   (b) $\theta_i'(z) > (=)\ 0$ *iff* $RAA_\phi(z) > (=)\ RAA_{\phi_i}$.

2. $RAA_\phi(z) = \left[\sum_i \theta_i(z) \frac{1}{\gamma_i}\right]^{-1} - \alpha$, *is strictly decreasing with z if* $\min_i \gamma_i < \max_i \gamma_i$ *that is, if relative ambiguity aversion is heterogeneous in the economy. It is constant if* $\min_i \gamma_i = \max_i \gamma_i$.

Part 1 shows that, θ_i, the slope coefficient of the linear sharing rule, is a function of the certainty equivalent of the aggregate consumption (in excess of ζ) and notes two properties of this function. Let i be more relatively ambiguity averse than j. Then, Part 1 (a) shows that i's share relative to j's decreases as we go to better models, that is, models with higher aggregate certainty equivalents. Hence, the more relatively ambiguity averse consumer has a smoother expected utility across models. Part 1 (b) shows that as we move from worse to better models, a consumer more (resp. less) relatively ambiguity averse than the representative consumer will see their

[22] The value function v corresponding to the Negishi outer program (9) is not independent of the weights (λ_i)'s since v_i's may not have CMRT. However, the property we establish about the representative consumer's relative ambiguity aversion holds irrespective of the specification of these weights.

share decrease (resp. increase) for models with marginally greater certainty equivalents. Finally, if relative ambiguity aversion were homogeneous, θ_i is a constant function. Hence, efficient allocations under ambiguity are different from those under expected utility *only* when ambiguity attitudes are heterogeneous.

Part 2 shows that the non-constant term in $RAA_{\phi}(z)$ is a weighted harmonic mean of the γ_i's, weighted by i's share of the aggregate certainty equivalent at an efficient allocation. Together with Part 1 (a), this implies that as we go to models with higher aggregate certainty equivalents, the representative consumer's relative ambiguity aversion is influenced more by consumers with lower relative ambiguity aversion. Thus, the relative ambiguity aversion of the representative consumer declines as models $\mathbf{P}$ get better. Remarkably, even though individual consumers have constant relative ambiguity aversion, at any efficient allocation the representative consumer has decreasing relative ambiguity aversion.[23]

Running Example continued. *At an efficient allocation, the share of endowments each consumer gets depends on whether the economy is in a Boom or in a bust and on the consumer's relative ambiguity aversion (if ambiguity aversion is heterogeneous). For any given pair of consumers, the ratio of the share of the more ambiguity averse to the share of the less ambiguity averse consumers is higher during a bust than during a Boom (this ratio would be constant if ambiguity aversion is homogeneous). This explains why efficient allocations are not comonotone. Suppose there are just two consumers, 1 and 2, with $\gamma_1 > \gamma_2$. According to Proposition 6,*

$$\frac{\theta_1(c_u^{\mathbf{P}_b}(\bar{x}))}{\theta_2(c_u^{\mathbf{P}_b}(\bar{x}))} > \frac{\theta_1(c_u^{\mathbf{P}_B}(\bar{x}))}{\theta_2(c_u^{\mathbf{P}_B}(\bar{x}))}.$$

[23]We extend the analysis to the cases of CARA u_i's and v_i's in Online-Appendix D –there, the representative consumer has constant relative ambiguity aversion. When u_i's and v_i's are quadratic, ϕ_i is linear and we are in a vNM-economy.

Since $\theta_1(c_u^{\mathbf{P}_b}(\bar{x})) = 1 - \theta_2(c_u^{\mathbf{P}_b}(\bar{x}))$, *this inequality implies that* $\theta_1(c_u^{\mathbf{P}_B}(\bar{x})) < \theta_1(c_u^{\mathbf{P}_b}(\bar{x}))$ *while the reverse inequality holds for consumer 2. Hence, consumer 1 has a higher consumption in the state* $\omega_{\mathbf{P}_b} \in \Omega_{\mathbf{P}_b}$ *than in* $\omega_{\mathbf{P}_B} \in \Omega_B$, *where* $\bar{X}(\omega_b) = \bar{X}(\omega_{\mathbf{P}_B}) = \bar{x}$, *while the reverse holds for consumer 2, establishing that the allocation is not comonotone.*

Note, to achieve efficiency in a market economy, consumers would have to be able to contract for two contingent amounts of a mutual fund, one contingent on the economy being in a bust next period, the other contingent on the economy being in a Boom. Thus, at an equilibrium allocation, relatively more of the resources will be allocated to relatively more ambiguity averse consumers in a bust than in a Boom, making the representative consumer more ambiguity averse in a bust.

2.3 Robustness to set-identifiability

(Denti & Pomatto 2020) provides an axiomatization for *partially-identifiable* preferences, where $\mathcal{P}$ is not point-identified but only set-identified, that is, the kernel is now set-valued and associates to each ω a set of probability laws the decision maker deems compatible with the observed evidence. The functional representing such preferences is

$$U_i(X_i) = \int_{\mathcal{M}} \phi_i\left(\min_{\mathbf{P}\in\mathbf{M}} E^{\mathbf{P}} u_i\left(X_i^{\mathbf{M}}\right)\right)\mu(d\mathbf{M}), \tag{10}$$

where $\mathcal{M}$ is the set of set-identified models, i.e., a collection of sets $\mathbf{M}$, each $\mathbf{M}$, being a set of probability laws. Within each $\mathbf{M} \in \mathcal{M}$, the decision maker is MEU and then aggregates, over $\mathbf{M} \in \mathcal{M}$, these utilities via the smooth ambiguity aggregator ϕ_i.

(Wakai 2007) showed that when consumers have MEU preferences satisfying Condition 1, the Pareto optimal allocations are comonotone and there is a representative consumer with HARA utility function (and the common marginal risk tolerance). Moreover, the consumers' utility are affinely related, which ensures that, within each set $\mathbf{M}$, at an efficient allocation, the (set of) minimizing prior(s) is the same for all individuals.[24] Armed with this result, we can "replace" each set-identified model, $\mathbf{M}$, by the worst probability law in $\mathbf{M}$, apply our analysis to this economy and obtain results analogous to those we have in the point-identified case. That is, Propositions 4 and 6 continue to hold, though efficient allocations are now contingent on $\mathbf{M} \in \mathcal{M}$ whereas previously they were contingent on $\mathbf{P} \in \mathcal{P}$. In this sense, key insights of our analysis of the representative consumer and sharing rule in economies satisfying Condition 1 remain robust to partial-identification, providing a further justification for our focus on this condition.

3 Pricing kernel

In this section, we explore the asset pricing implications of heterogeneous ambiguity aversion. Specifically, we use the features of the representative agent described in Proposition 6 to derive the properties of the pricing kernel. The macro-finance literature that has used the smooth ambiguity model (for instance, (Ju & Miao 2012), (Collard et al. 2018), (Hansen & Miao 2018), (Hansen & Miao 2022), (Gallant, Jahan-Parvar & Liu 2019), and (Thimme & Volkert 2015)) assumed constant relative ambiguity aversion for the representative consumer. As we showed, this corresponds to homogeneity of ambiguity aversion in the underlying economy. Relaxing this assumption and allowing for heterogeneity, i.e., a decreasing relative ambiguity aversion representative consumer, yield new results, closer to documented empirical regularities of the pricing kernel. While the cited literature already shows

[24]One may define $\mathbf{M}$-conditionally efficient allocations analogous to the definition of $\mathbf{P}$-conditionally efficient allocations for the point-identified case. More precisely, Wakai's result ensures that at an $\mathbf{M}$-conditionally efficient allocation, consumers' utilities are affinely related.

that ambiguity aversion allows equity premium to be high enough to match data, our analysis shows that, in addition, ambiguity aversion causes the equity premium to be counter-cyclical.

As explained in footnote 8, under identifiability and model-independence of the range of aggregate endowment, there exists a coarsening of Ω that can be written as $\bar{X}(\Omega) \times \mathcal{P}$. This coarsening is enough for our purposes. We assume that $\bar{X}(\Omega) = \mathbb{R}_+$ and denote a generic element by x.[25] Let $P(x) = \mathbf{P}(\{\omega \in \Omega | \bar{X}(\omega) \leq x\})$ be the cumulative distribution function of endowment realization under model $\mathbf{P}$ and let $p(x)$ be the associated density with respect to the Lebesgue measure.

Running Example continued. *In (Cecchetti, Lam & Mark 1990) the specification for the probability distributions of x in the two regimes are log-normals with a common variance, with bust having a lower mean than Boom. (Kandel & Stambaugh 1991) additionally allow the two regimes to have different variances. (Ju & Miao 2012), in their dynamic Lucas tree model, also use such a Markovian endowment process. Note that modeling the endowment process in a dynamic Lucas tree in this way implies that the ambiguous uncertainty about the regimes is renewed every period as time goes by.*

The tuple (μ, ϕ, u, v) describes components of the representative consumer's smooth ambiguity preferences with $\phi \circ u = v$. We derive properties of asset prices assuming a complete set of securities. The price density supporting the aggregate endowment as an equilibrium of the representative consumer economy is given by a function $\psi : \bar{X}(\Omega) \times \mathcal{P} \to \mathbb{R}_{++}$ such that

$$\psi(x, \mathbf{P}) = \phi' \left(E^{\mathbf{P}} u \left(\bar{X}^{\mathbf{P}} \right) \right) p(x) u' \left(\bar{X}^{\mathbf{P}}(x) \right). \tag{11}$$

An x-contingent claim delivers a unit of the good if x occurs, no matter what $\mathbf{P}$ is, and hence its price density is the sum, over models in $\mathcal{P}$, of the price

[25] Note that here, we take $\bar{X}(\Omega)$, and hence Ω, to be infinite.

densities of $(x, \mathbf{P})$-contingent claims:

$$\int_{\mathcal{P}} \phi'\left(E^{\mathbf{P}} u\left(\bar{X}\right)\right) p(x) u'(x) \mu(d\mathbf{P}). \tag{12}$$

Divide this price by the density of x with respect to the reduced measure, $p^{\mu}(x) = \int_{\mathcal{P}} q(x)\mu(d\mathbf{Q})$, to obtain the *pricing kernel*:[26]

$$x \mapsto \pi_{u,\phi}(x) \equiv \int_{\mathcal{P}} \frac{p(x)}{p^{\mu}(x)} \phi'\left(E^{\mathbf{P}} u\left(\bar{X}\right)\right) u'(x)\mu(d\mathbf{P}). \tag{13}$$

The pricing kernel under ambiguity aversion is a weighted average of marginal utilities with weights $\frac{p(x)}{p^{\mu}(x)}\phi'\left(E^{\mathbf{P}}u(\bar{X})\right)$ whereas under ambiguity neutrality it is simply the marginal utility. It allows us to price any contingent claim written on aggregate endowment. Let $y : \mathbb{R}_{++} \to \mathbb{R}$ be such a claim. Then, the price of y is equal to

$$E\left[\pi_{u,\phi} y\right] = \int_{\mathbb{R}_{++}} \pi_{u,\phi}(x) y\left(x\right) p^{\mu}(x) dx. \tag{14}$$

The *elasticity of the pricing kernel* $\pi_{u,\phi}$ at x, given by $\varepsilon(x; \pi_{u,\phi}) \equiv -\frac{\pi'_{u,\phi}(x)x}{\pi_{u,\phi}(x)}$, a measure of the kernel's variability. The Hansen-Jagannathan (H-J) bound of the pricing kernel $\pi_{u,\phi}$ is equal to $\sigma\left[\pi_{u,\phi}\right]/E\left[\pi_{u,\phi}\right]$. It is, in principle, deducible from returns data. A theory that delivers a higher H-J bound has a greater potential to accommodate market volatility and large equity premia. In what follows, we give two illustrations of how the elasticity of the pricing kernel and the H-J bound are affected by ambiguity aversion. In the first, we assume a Gaussian environment which allows us to obtain an analytical characterization. In the second, we return to the running example.

Assumption 1.

1. *For each* $\mathbf{P}$, $\bar{X}^{\mathbf{P}}$ *is log-normally distributed,* $log(\bar{X}^{\mathbf{P}}) \sim \mathcal{N}(m_{\mathbf{P}}, \sigma^2)$ *and* $\mathcal{P}$ *is the set of all such log-normal distributions.*

[26]If the representative consumer were both risk and ambiguity neutral, he price density in 12 would reduce to $P^{\mu}(x)$.

2. *The prior on the parameters $m_{\mathbf{P}} \in \mathbb{R}$ is $\mathcal{N}(\hat{m}, \hat{\sigma}^2)$.*

It is common in the macro-finance literature to assume that the aggregate consumption is log-normal, and part 1 of the above assumption follows that practice and allows for uncertainty about the mean growth parameter. The two parts, taken together, ensure that an ambiguity neutral consumer believes that aggregate consumption is log-normal.

Proposition 7. *Suppose Assumption 1 holds, u is CRRA and $\phi''(.) < 0$.*

1. *If RAA_{ϕ} is constant, then $\varepsilon(x; \pi_{u,\phi})$ is constant.*

2. *If RAA_{ϕ} is strictly decreasing, then $\varepsilon(x; \pi_{u,\phi})$ is strictly decreasing in x.*

Proposition 7 taken together with Proposition 6, allows us to compare properties of the pricing kernel across two economies: one with homogeneous relative ambiguity aversion and the other with heterogeneous relative ambiguity aversion. In part 1, let the ϕ correspond to a homogeneous multi-consumer economy where $u_i = u$ and $v_i = v$ with the CRRA coefficients α and γ respectively. Then, the elasticity of the pricing kernel is constant and, in fact, may be described explicitly: for every $x > 0$,

$$\varepsilon(x, \pi_{u;\phi}) = \frac{\sigma^2}{\sigma^2 + \hat{\sigma}^2}\alpha + \frac{\hat{\sigma}^2}{\sigma^2 + \hat{\sigma}^2}\gamma. \tag{15}$$

It is evident from (15) that, ceteris paribus, the higher the consumer's ambiguity aversion $\gamma - \alpha$, or the higher the ambiguity, in the sense of a larger $\hat{\sigma}^2$, the higher the elasticity of the kernel, as illustrated in the left panel of Figure 1. In Part 2, the ϕ corresponds to a heterogeneously ambiguity averse economy. We compare the pricing kernel in such an economy with that in an economy as in Part 1. Assume the relative ambiguity aversion in the homogeneous economy lies strictly between the maximum and minimum relative ambiguity aversion in the heterogeneously ambiguity averse economy

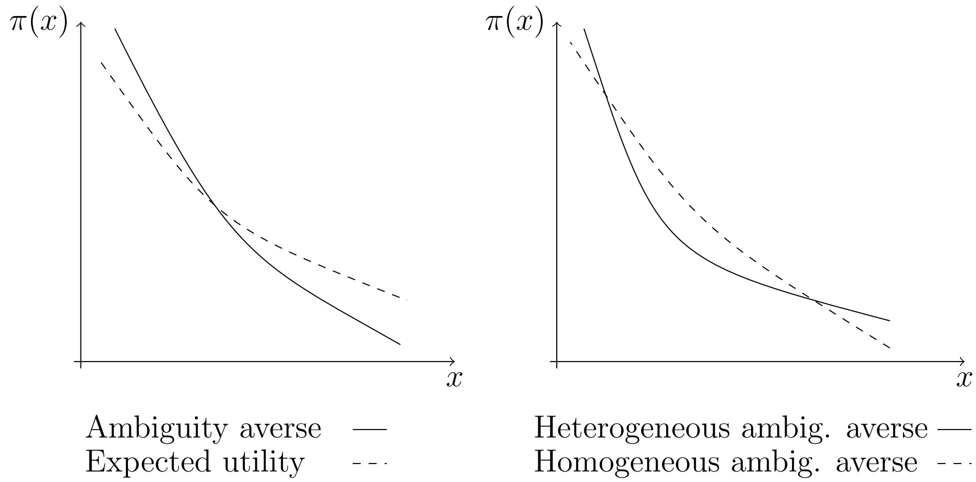

Figure 1: Pricing kernels

and normalize the pricing kernels so that the two economies have the same risk-free rate. Then, Proposition 7 (part 2) implies that the kernels have exactly two points of intersection and have the distinct qualitative features depicted in the right panel of Figure 1. [27] For low values of (aggregate) consumption the kernel of the heterogeneously ambiguity averse economy will be more elastic and, thus, steeper than that of the homogeneously ambiguity averse economy. For high values of aggregate consumption, the relation between the slopes of the two kernels is the other way round. Under heterogeneous ambiguity aversion the pricing kernel is more elastic in "bad times" compared to "good times", a point we formalize in the following proposition.

To state the next proposition, we write $\pi_{u,\phi}(x, \hat{m})$ instead of $\pi_{u,\phi}(x)$, and $E^{\hat{m}}$ and $\sigma^{\hat{m}}$ instead of E and σ, to make explicit the dependence of the pricing kernel and H-J bound on $\hat{m}$, the mean of the second order beliefs.

Proposition 8. *Suppose Assumption 1 holds, u is CRRA and $\phi''(.) < 0$.*

1. *If RAA_ϕ is constant, then* $\dfrac{\sigma^{\hat{m}}(\pi_{u,\phi}(., \hat{m}))}{E^{\hat{m}}(\pi_{u,\phi}(., \hat{m}))}$ *is constant in $\hat{m}$.*

[27] The details of the argument can be found in Proposition 9 in the Appendix.

2. If RAA_ϕ is strictly decreasing then $\dfrac{\sigma^{\hat{m}}(\pi_{u,\phi}(.,\hat{m}))}{E^{\hat{m}}(\pi_{u,\phi}(.,\hat{m}))}$ *is strictly decreasing in $\hat{m}$.*

To see the implications of the second part, consider two scenarios, 1 and 2, with $\hat{m}_2 > \hat{m}_1$. We interpret scenario 2 as one where a typical consumer views the immediate future more optimistically relative to scenario 1. Under this interpretation, the result shows that the H-J bound is *counter-cyclical* if there is heterogeneity in the relative ambiguity aversion of the consumers in the underlying economy, whereas it is *constant* across the cycle if relative ambiguity aversion is homogeneous. If markets are complete, as we have assumed, the H-J bound equals the highest Sharpe ratio that can be achieved by any portfolios of assets. Our result is empirically compelling since the Sharpe ratio for U.S. aggregate stock market is significantly counter-cyclical and volatile. [28]

In Assumption 1 the volatility is held constant across models. If we relaxed this assumption, we can show numerically that the pricing kernel might have an upward sloping segment. We do this using our running example.

Running Example continued. *Let P_B and P_b be two log-normals such that P_B has a high mean and low variance, while P_b has a low mean and a high variance.*[29] *The specification has the feature that the conditional likelihood of the bust model may* increase *given an increase in x, a realisation of $\bar{X}$. Since this regime is associated with a lower expected utility, this could lead to an increase of the first part of the weighted average* $\left[\frac{p_b(x)}{p^\mu(x)}\phi'\left(E^{\mathbf{P}_b}u\left(\bar{X}\right)\right) + \frac{p_B(x)}{p^\mu(x)}\phi'\left(E^{\mathbf{P}_B}u\left(\bar{X}\right)\right)\right]u'(x)$ *that appears in (13). On the other hand, an increase in x means a lower second component, that is,*

[28] (Rosenberg & Engle 2002) obtain a measure of "empirical risk aversion" using the risk aversion implied by the pricing kernel they estimate. They show that this risk aversion varies counter-cyclically, supporting earlier results of (Fama & French 1989) who showed that risk premia are negatively correlated with the business cycle. See also (Lettau & Ludvigson 2010).

[29] The parameter values are estimates based on historical data, see Appendix.

a lower marginal utility $u'(x)$. Thus, risk aversion and ambiguity aversion drive the kernel in opposite directions. If ambiguity aversion dominates for a range of endowment, then the state price instead of falling with an increase in x turns up, giving the pricing kernel a positive slope (locally).

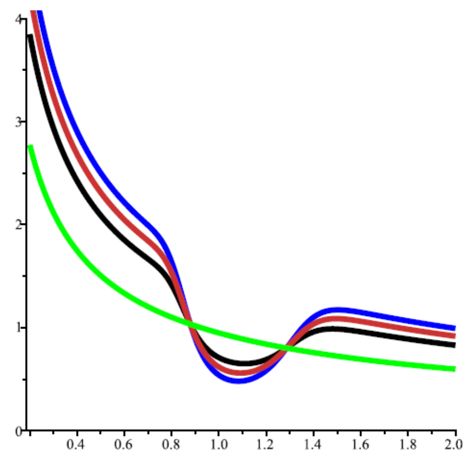

Figure 2: Graphs of the pricing kernels in this example for four economies. u is CRRA with $\alpha = 2/3$. (1) — : representative vNM consumer economy. (2) — : representative smooth ambiguity consumer economy, v is CRRA, $\gamma = 12$. (3) — : representative smooth ambiguity consumer economy, v is CRRA, $\gamma = 6$. (4) — : economy constituted by 2 consumers, one of whom has preferences as in (2) and the other as in (3).

This kind of non-monotonicity, anticipated in (Gollier 2011), provides an explanation of the so-called *pricing kernel puzzle,* discussed in (Hens & Reichlin 2013) and (Cuesdeanu & Jackwerth 2018). The puzzle refers to the empirical evidence that the downward slope of the pricing kernel implied by a risk averse EU representative consumer is violated in reality: there is an interval, away from extreme (negative or positive) returns, where the pricing kernel is increasing, as in Figure 5 of (Rosenberg & Engle 2002) and our Figure 2.

Proofs

Proofs of Propositions in Section 1

Proof of Proposition 2 (Part 2.) At a Pareto optimum, if $\bar{X}^{\mathbf{P}}(\omega) = \bar{X}^{\mathbf{P}}(\omega')$ for some $\omega, \omega' \in \Omega_{\mathbf{P}}$ then $X_i^{\mathbf{P}}(\omega) = X_i^{\mathbf{P}}(\omega')$ for all i (this is a direct consequence of the strict concavity of the Bernoulli utility function). Hence, we can write $X_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = X_i^{\mathbf{P}}(\omega)$ for any $x \in \bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})$ and $\omega \in \Omega_{\mathbf{P}}$ with $\bar{X}^{\mathbf{P}}(\omega) = x$. Since $\bar{X}$ is unambiguous, $\mathbf{P}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = \mathbf{Q}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right) \equiv \zeta(x)$ for all $x \in \bar{X}(\Omega)$. (Recall, $\bar{X}(\Omega) = \bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}}) = \bar{X}^{\mathbf{Q}}(\Omega_{\mathbf{Q}})$.)

(only if) Let $Y = (Y^{\mathbf{P}})_{\mathbf{P}}$ be an efficient allocation and, thus, conditionally efficient. Assume there exist i, $x \in \bar{X}(\Omega)$ and $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ such that $Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) \neq Y_i^{\mathbf{Q}}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right)$. For any i and any $x \in \bar{X}(\Omega)$, define $Y_i^{\star} : \Omega \to \mathbb{R}_+$ so that: $Y_i^{\star}((\bar{X})^{-1}(x)) = \sum_{\mathbf{P}\in\mathcal{P}} \mu(P) Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)$. $Y^{\star}$ is feasible:

$$\begin{aligned}
\sum_i Y_i^{\star}((\bar{X})^{-1}(x)) &= \sum_i \sum_{\mathbf{P}} \mu(\mathbf{P}) Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) \\
&= \sum_{\mathbf{P}} \mu(\mathbf{P}) \sum_i Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) \\
&= \sum_{\mathbf{P}} \mu(\mathbf{P}) x = x
\end{aligned}$$

Next, we show that $Y^\star$ Pareto dominates the allocation Y.

$$\begin{aligned}
U_i(Y_i) \quad &= \quad \sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\left(E^{\mathbf{P}} u_i\left(Y_i^{\mathbf{P}}\right)\right) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\Big(\sum_{\omega\in\Omega_{\mathbf{P}}} \mathbf{P}(\omega) u_i\left(Y_i^{\mathbf{P}}(\omega)\right)\Big) \\
&= \quad \sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\Big(\sum_{x\in\bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})} \mathbf{P}((\bar{X}^{\mathbf{P}})^{-1}(x)) u_i\left(Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\right)\Big) \\
&\le (<) \quad \phi_i\Big(\sum_{\mathbf{P}} \mu(\mathbf{P}) \sum_{x\in\bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})} \zeta(x) u_i\left(Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\right)\Big) \\
&\le (<) \quad \phi_i\Big(\sum_{x\in\bar{X}(\Omega)} \zeta(x) u_i\Big(\sum_{\mathbf{P}} \mu(\mathbf{P}) Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\Big)\Big) \\
&= \quad \phi_i\Big(\sum_{x\in\bar{X}(\Omega)} \zeta(x) u_i\left(Y_i^\star\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right)\right)\Big) = U_i(Y_i^\star)
\end{aligned}$$

Note, some weak inequalities in the derivation above are strict for at least one i. Hence, $Y^\star$ Pareto dominates Y, a contradiction.

(if) Let $Y = (Y^{\mathbf{P}})_{\mathbf{P}}$ be a conditionally efficient allocation such that $Y_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = Y_i^{\mathbf{Q}}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right)$ for all i, all $\mathbf{P}$, $\mathbf{Q}$ and all $x \in \bar{X}(\Omega)$. Assume it is not efficient. Then, there exists an efficient allocation $\hat{Y} = (\hat{Y}^{\mathbf{P}})_{\mathbf{P}}$ that Pareto dominates it. By the same argument as in the (only if) part of the proof, $\hat{Y}_i^{\mathbf{P}}\left((\bar{X}^{\mathbf{P}})^{-1}(x)\right) = \hat{Y}_i^{\mathbf{Q}}\left((\bar{X}^{\mathbf{Q}})^{-1}(x)\right)$ for all i, all $\mathbf{P}$, $\mathbf{Q}$ and all $x \in \bar{X}(\Omega)$. Since endowment is unambiguous, we have that $E^{\mathbf{P}} u_i(\hat{Y}_i^{\mathbf{P}}) = E^{\mathbf{Q}} u_i(\hat{Y}_i^{\mathbf{P}})$ for all $i, \mathbf{P}, \mathbf{Q}$.
Therefore, $\sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right) = \phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right)$ for some (any) $\mathbf{P}$. The same holds for Y and hence, $\sum_{\mathbf{P}} \mu(\mathbf{P})\phi_i\left(E^{\mathbf{P}}(u_i(Y_i^{\mathbf{P}}))\right) = \phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right)$ for any $\mathbf{P}$. That $\hat{Y}$ Pareto dominates Y therefore means that $\phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right) \ge \phi_i\left(E^{\mathbf{P}}(u_i(\hat{Y}_i^{\mathbf{P}}))\right)$ for all i with a strict inequality for at least one. But this is a contradiction to the fact that Y is conditionally efficient. □

Proof of Proposition 3 The necessary and sufficient condition for efficiency of an interior allocation $(X_i)_{i=1,\ldots,I}$ is that for all $\mathbf{P} \in \mathcal{P}$, there exists

$\psi^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{R}_{++}$ s.th. $\forall i$, $\exists \lambda_i > 0$ s.th. $\forall \omega_{\mathbf{P}} \in \Omega_{\mathbf{P}}$,

$$\psi^{\mathbf{P}}(\omega_{\mathbf{P}}) = \lambda_i \phi_i' \left(E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}}\right)\right) u_i' \left(X_i^{\mathbf{P}}(\omega)\right). \tag{16}$$

Since $\bar{X}$ is unambiguous $\bar{X}(\Omega) = \bar{X}^{\mathbf{P}}(\Omega_{\mathbf{P}})$ for all $\mathbf{P}$. For all $x \in \bar{X}(\Omega)$ and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$:

$$\frac{\psi^{\mathbf{Q}}(\omega_{\mathbf{Q}})}{\psi^{\mathbf{P}}(\omega_{\mathbf{P}})} = \frac{\phi_i' \left(E^{\mathbf{Q}} u_i \left(X_i^{\mathbf{Q}}\right)\right) u_i' \left(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{\phi_i' \left(E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}}\right)\right) u_i' \left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \quad \forall i. \tag{17}$$

Let κ be s.th.

$$\frac{\phi_\kappa' \left(E^{\mathbf{Q}} u_\kappa \left(X_\kappa^{\mathbf{Q}}\right)\right)}{\phi_\kappa' \left(E^{\mathbf{P}} u_\kappa \left(X_\kappa^{\mathbf{P}}\right)\right)} \geq \frac{\phi_i' \left(E^{\mathbf{Q}} u_i \left(X_i^{\mathbf{Q}}\right)\right)}{\phi_i' \left(E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}}\right)\right)} \quad \forall i = 1 \ldots, I.$$

To simplify exposition, let $\kappa = 1$. By (17), for all x and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$:

$$\frac{u_1' \left(X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_1' \left(X_1^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \leq \frac{u_i' \left(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_i' \left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \quad \forall i.$$

If $X_1^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})$, then the l.h.s. is strictly greater than one. Hence $X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for every i, a contradiction to $\sum_i X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) = x = \sum_i X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$. Hence, $X_1^{\mathbf{P}}(\omega_{\mathbf{P}}) \leq X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for every x, $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$.

Since $u_1' > 0$, $E^{\mathbf{P}} u_1 \left(X_1^{\mathbf{P}}\right) \leq E^{\mathbf{P}} u_1 \left(X_1^{\mathbf{Q}}\right)$. Since $X_1^{\mathbf{Q}}$ is strictly monotone in $\bar{X}$, $\mathbf{P} \circ \left(X_1^{\mathbf{Q}}\right)^{-1}$ is FOS dominated by $\mathbf{Q} \circ \left(X_1^{\mathbf{Q}}\right)^{-1}$. Thus, $E^{\mathbf{P}} u_1 \left(X_1^{\mathbf{Q}}\right) \leq E^{\mathbf{Q}} u_1 \left(X_1^{\mathbf{Q}}\right)$. Hence, $E^{\mathbf{P}} u_1 \left(X_1^{\mathbf{P}}\right) \leq E^{\mathbf{Q}} u_1 \left(X_1^{\mathbf{Q}}\right)$. Thus,

$$\frac{\phi_i' \left(E^{\mathbf{Q}} u_i \left(X_i^{\mathbf{Q}}\right)\right)}{\phi_i' \left(E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}}\right)\right)} \leq \frac{\phi_1' \left(E^{\mathbf{Q}} u_1 \left(X_1^{\mathbf{Q}}\right)\right)}{\phi_1' \left(E^{\mathbf{P}} u_1 \left(X_1^{\mathbf{P}}\right)\right)} \leq 1 \quad \forall i = 1, \ldots, I. \tag{18}$$

Since $\frac{u_1'\left(X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_1'\left(X_1^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \leq 1$ for every x and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$, by (17) and (18),

$$\frac{\psi^{\mathbf{Q}}(\omega_{\mathbf{Q}})}{\psi^{\mathbf{P}}(\omega_{\mathbf{P}})} = \frac{\phi_1' \left(E^{\mathbf{Q}} u_1 \left(X_1^{\mathbf{Q}}\right)\right) u_1' \left(X_1^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{\phi_1' \left(E^{\mathbf{P}} u_1 \left(X_1^{\mathbf{P}}\right)\right) u_1' \left(X_1^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} \leq 1. \tag{19}$$

To show that $E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}} \right) \leq E^{\mathbf{Q}} u_i \left(X_i^{\mathbf{Q}} \right)$, consider two cases:

- If $X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) \leq X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$ for all x and all $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$ and $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$, then we can show that $E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}} \right) \leq E^{\mathbf{P}} u_i \left(X_i^{\mathbf{Q}} \right) \leq E^{\mathbf{Q}} u_i \left(X_i^{\mathbf{Q}} \right)$.
- If not, there are an x, an $\omega_{\mathbf{P}} \in (\bar{X}^{\mathbf{P}})^{-1}(x)$, and an $\omega_{\mathbf{Q}} \in (\bar{X}^{\mathbf{Q}})^{-1}(x)$ s.th. $X_i^{\mathbf{P}}(\omega_{\mathbf{P}}) > X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})$. For such an $(x, \omega_{\mathbf{P}}, \omega_{\mathbf{Q}})$, $\frac{u_i'\left(X_i^{\mathbf{Q}}(\omega_{\mathbf{Q}})\right)}{u_i'\left(X_i^{\mathbf{P}}(\omega_{\mathbf{P}})\right)} > 1$. By (19), $\frac{\phi_i'\left(E^{\mathbf{Q}} u_i\left(X_i^{\mathbf{Q}}\right)\right)}{\phi_i'\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right)} < 1$. Since $\phi_i'' < 0$, $E^{\mathbf{P}} u_i \left(X_i^{\mathbf{P}} \right) < E^{\mathbf{Q}} u_i \left(X_i^{\mathbf{Q}} \right)$.

□

Proofs of Propositions in Section 2

Proof of Lemma 1 Focus first on a vNM-economy where consumers are CARA with parameter α_i. The representative consumer has then CARA utility with parameter α, $\sum_i (\alpha/\alpha_i) = 1$, and the sharing rule is $X_i = (\alpha/\alpha_i)\bar{X} + \tau_i$ where $\sum_i \tau_i = 0$. Direct computation yields

$$u_i^{-1} \left(E^{\mathbf{P}} u_i \left(X_i \right) \right) = \frac{\alpha}{\alpha_i} u^{-1} \left(E^{\mathbf{P}} u \left(\bar{X} \right) \right) + \tau_i. \tag{20}$$

Hence, $\sum_i u_i^{-1} \left(E^{\mathbf{P}} u_i(X_i) \right) = u^{-1} \left(E^{\mathbf{P}} u(\bar{X}) \right)$.

Consider next the case where consumers have non-zero CMRT, $u_i(x_i) = \frac{\alpha}{1-\alpha} \left(\frac{x_i - \zeta_i}{\alpha} \right)^{1-\alpha}$ for $\alpha \neq 0$ and $\alpha \neq 1$.[30]

The representative consumer has utility $u(x) = \frac{\alpha}{1-\alpha} \left(\frac{x-\zeta}{\alpha} \right)^{1-\alpha}$, where $\zeta = \sum_i \zeta_i$. The sharing rule is $X_i = \theta_i(\bar{X} - \zeta) + \zeta_i$ where $\sum_i \theta_i = 1$. Direct calculation yields $u_i^{-1} \left(E^{\mathbf{P}} u_i(X_i) \right) = \theta_i \left(u^{-1} \left(E^{\mathbf{P}} u(X) \right) - \zeta \right) + \zeta_i$ leading to $\sum_i u_i^{-1} \left(E^{\mathbf{P}} u_i \left(X_i \right) \right) = u^{-1} \left(E^{\mathbf{P}} u \left(\bar{X} \right) \right)$. □

Proof of Proposition 4 Beyond the arguments in the text, it remains to prove that $[\exists i \text{ s.th. } \phi_i'' < 0] \Rightarrow \phi'' < 0$. Denote risk tolerance of u at x by $ART(x; u)$. Then, $ART(x, u) = \sum_i ART(g_i(x), u_i)$ where x is consumption

[30]The HARA with CMRT family also includes $u_i(X_i) = \ln(X_i - \zeta_i)$.

levels and $g_i(x)$ a solution to (8) (Wilson 1968). Similarly, $ART(x,v) = \sum_i ART(f_i(x),v_i)$, where x represents certainty equivalents and $f_i(x)$ the solution to (9). Note $\sum_i g_i(x) = x = \sum_i f_i(x)$. As the u_i's are HARA with CMRT, if $ART(x_i,u_i) = a_i + bx$, then:

$$\begin{aligned}\sum_i ART(g_i(x),u_i) = & \quad \sum_i a_i + b\sum_i g_i(x) \\ = & \quad \sum_i a_i + b\sum_i f(x_i) = \quad \sum_i ART(f_i(x),u_i)\end{aligned}$$

Since v_i is (strictly) more concave than u_i for all (at least one) i,

$$\sum_i a_i + \sum_i ART(f_i(x),u_i) > \sum_i a_i + \sum_i ART(f_i(x),v_i) = ART(x;v).$$

Hence, $ART(x;u) > ART(x;v)$ for all x, that is v is more concave than u, i.e., $\phi'' < 0$. EU-comonotonicity comes from comonotonicity of efficient allocations in vNM-economies applied to (9). □

Proof of Proposition 5 See Online Appendix C □

Proof of Proposition 6

1. Each **P**-conditionally efficient allocation can be written $X_i^{\mathbf{P}} = \theta_i^{\mathbf{P}}(\bar{X} - \zeta) + \zeta_i$ for some $\theta_i^{\mathbf{P}}$ (Section 3.6, (Back 2017)). We prove that $\theta_i^{\mathbf{P}}$ is a function of $c_u^{\mathbf{P}}$. Efficient allocations can be obtained by solving (9), for some $(\lambda_i^{\mathbf{P}})_i$, to allocate aggregate certainty equivalents $c^{\mathbf{P}}$ under each model **P**.

For each $c > \zeta$, let $(\hat{f}_i(c))_i$ be the solution to (9). Then, $\hat{f}_i(c_u^{\mathbf{P}}) = u_i^{-1}\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right)$. For each $z > 0$, define $f_i(z) = \hat{f}_i(z+\zeta) - \zeta_i$ and $\theta_i(z) = f_i(z)/z$. Then $u_i^{-1}\left(E^{\mathbf{P}} u_i\left(X_i^{\mathbf{P}}\right)\right) = \theta_i(c_u^{\mathbf{P}} - \zeta) \times (c_u^{\mathbf{P}} - \zeta) + \zeta_i$. Since v is the value function of (9), the envelope theorem implies that $\lambda_i v_i'(f_i(z)+\zeta_i) = v'(z+\zeta)$. Hence, $v'(z+\zeta) > 0$. Differentiating w.r.t. z, $\lambda_i v_i''(f_i(x)+\zeta_i) f_i'(z) = v''(z+\zeta)$. Dividing each side of the second equality by the corresponding side of the first :

$$-\frac{v_i''(f_i(z)+\zeta_i)f_i(z)}{v_i'(f_i(z)+\zeta_i)}\frac{f_i'(z)}{f_i(z)} + \frac{v''(z+\zeta)}{v'(z+\zeta)} = 0$$

for every i and $z > 0$. Since v_i exhibits HARA with parameters (γ_i, ζ_i),

$$\gamma_i \frac{f_i'(z)}{f_i(z)} + \frac{v''(z+\zeta)}{v'(z+\zeta)} = 0. \tag{21}$$

Since $\sum_i f_i(z) = z$ $\exists i$ s.th. $f_i'(z) > 0$. Thus, $v''(z+\zeta) < 0$. Hence, $f_i'(z) > 0$ for every i. Moreover,

$$\frac{d}{dz} \ln \frac{f_j(z)}{f_i(z)} = \frac{f_j'(z)}{f_j(z)} - \frac{f_i'(z)}{f_i(z)} = -\frac{v''(z+\zeta)}{v'(z+\zeta)} \left(\frac{1}{\gamma_j} - \frac{1}{\gamma_j} \right) \gtreqless 0$$

if and only if $\gamma_i \gtreqless \gamma_j$. Since $f_j(z)/f_i(z) = \theta_j(z)/\theta_i(z)$, the sign property in part 1 (a) is proved. Part 1 (b) follows from (24) below.

2. (Corollary 7 part 2 of (Hara, Huang & Kuzmics 2007)). One can write (21) as $\theta_i(z)/\gamma_i = f_i'(z)/b(z)$. Hence,

$$\sum_i \theta_i(z) \frac{1}{\gamma_i} = \frac{1}{b(z)}. \tag{22}$$

Differentiating w.r.t. z, $\sum_i \theta_i'(z) \frac{1}{\gamma_i} = -\frac{b'(z)}{(b(z))^2}$. Since $\sum_i \theta_i'(z) = 0$,

$$\sum_i \theta_i'(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)} \right) = -\frac{b'(z)}{(b(z))^2}. \tag{23}$$

Since

$$\theta_i'(z) = \frac{\theta_i(z)}{z} \left(\frac{b(z)}{\gamma_i} - 1 \right), \tag{24}$$

we have

$$\sum_i \theta_i'(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)} \right) = \frac{b(z)}{z} \sum_i \theta_i(z) \left(\frac{1}{\gamma_i} - \frac{1}{b(z)} \right)^2.$$

If $\min_i \gamma_i < \max_i \gamma_i$, by (22), $\exists i$ s.th. $1/\gamma_i < 1/b(z)$ (and another i s.th. $1/\gamma_i > 1/b(z)$). Thus, the r.h.s. is strictly positive. By (23), $b'(z) < 0$. If $\min_i \gamma_i = \max_i \gamma_i$, then, by (22), $1/\gamma_i = 1/b(z)$ $\forall i$. By (23), $b'(z) = 0$. □

Proofs of Propositions in Section 3

Lemma 2 establishes results used in the proofs of Propositions 7 and 8.

Lemma 2. *1. Let P be any non-degenerate probability on $\mathbb{R}_{++}$. For $n = 1, 2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be continuous. Assume π_2 is non-increasing and π_2/π_1 is strictly increasing. Then, $\sigma(\pi_1)/E(\pi_1) > \sigma(\pi_2)/E(\pi_2)$, where E and σ are the mean and standard deviation under P.*

2. For $n = 1, 2$, let P_n be any non-degenerate probability on $\mathbb{R}_{++}$. Assume that P_n has a probability density function g_n and that there is a $k > 1$ such that $g_1(x) = kg_2(kx)$ for every $x > 0$. Let $\pi : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be differentiable. Assume that $\pi' < 0$ and $-\pi'(x)x/\pi(x)$ is strictly decreasing in x. Then $\sigma^{P_1}(\pi)/E^{P_1}(\pi) > \sigma^{P_2}(\pi)/E^{P_2}(\pi)$, where, for each n, E^{P_n} and σ^{P_n} are the mean and standard deviation under P_n.

Proof of Lemma 2

1. For each n, the integral of the function $x \mapsto (E(\pi_n))^{-1}\pi_n(x)$ under P is equal to one. Since it is continuous, (the graphs of) these two functions $n = 1, 2$ cross at least once. Since π_2/π_1 is strictly increasing, they cross exactly once. Let x^* be such that $\pi_1(x^*)/E(\pi_1) = \pi_2(x^*)/E(\pi_2)$ and denote this value by z^*. Then $\pi_1(x)/E(\pi_1) \gtreqqless \pi_2(x)/E(\pi_2)$ if and only if $x \lesseqqgtr x^*$. Since π_2 is non-increasing,

$$\frac{\pi_1(x)}{E(\pi_1)} \gtreqqless \frac{\pi_2(x)}{E(\pi_2)} \gtreqqless z^* \text{ if and only if } x \lesseqqgtr x^*.$$

Thus, for every $x \neq x^*$, $\left(\frac{\pi_1(x)}{E(\pi_1)} - z^*\right)^2 > \left(\frac{\pi_2(x)}{E(\pi_2)} - z^*\right)^2$. If $x = x^*$, then this inequality would hold as an equality. Since P is not degenerate,

$$\int \left(\frac{\pi_2(x)}{E(\pi_2)} - z^*\right)^2 P(dx) > \int \left(\frac{\pi_1(x)}{E(\pi_1)} - z^*\right)^2 P(dx).$$

Note that, for each $n = 1, 2$,

$$\begin{aligned}\frac{\sigma(\pi_n)^2}{E(\pi_n)^2} &= \int \left(\frac{\pi_n(x)}{E(\pi_n)} - 1\right)^2 P(dx) \\ &= \int \left(\left(\frac{\pi_n(x)}{E(\pi_n)} - z^*\right) + (z^* - 1)\right)^2 P(dx) \\ &= \int \left(\frac{\pi_n(x)}{E(\pi_n)} - z^*\right)^2 P(dx) - (z^* - 1)^2 .\end{aligned}$$

Thus, $\sigma(\pi_1)^2/E(\pi_1)^2 > \sigma(\pi_2)^2/E(\pi_2)^2$. Thus, $\sigma(\pi_1)/E(\pi_1) > \sigma(\pi_2)/E(\pi_2)$.
2. We prove this part by applying part 1. To do so, write P for P_1, g for g_1, and π_1 for π. Define π_2 by letting $\pi_2(x) = \pi_1(kx)$ for every $x \in I$. We now show that $E^P(\pi_2) = E^{P_2}(\pi)$ and $\sigma^P(\pi_2) = \sigma^{P_2}(\pi)$. Since $g_1(x) = kg_2(kx)$, the change-of-variable formula implies that

$$E^P(\pi_2) = \int \pi_2(x)g_1(x)dx = \int \pi_2(kx)kg_2(kx)dx = \int \pi(x)f_2(x)dx = E^{P_2}(\pi).$$

By this equality and the change-of-variables formula,

$$\begin{aligned}\sigma^P(\pi_2) &= \left(\int \left(\pi_2(x) - E^P(\pi_2)\right)^2 g_1(x)dx\right)^{1/2} \\ &= \left(\int \left(\pi(kx) - E^{P_2}(\pi)\right)^2 kg_2(kx)dx\right)^{1/2} \\ &= \left(\int \left(\pi(x) - E^{P_2}(\pi)\right)^2 g_2(x)dx\right)^{1/2} = \sigma^{P_2}(\pi).\end{aligned}$$

Thus, $\sigma^P(\pi_2)/E^P(\pi_2) = \sigma^{P_2}(\pi)/E^{P_2}(\pi)$.

It is thus enough to show $\sigma^P(\pi_1)/E^P(\pi_1) > \sigma^P(\pi_2)/E^P(\pi_2)$. By part 1, it suffices to prove that π_2/π_1 is strictly increasing. Differentiate both sides of $\pi_2(x) = \pi_1(kx)$ with respect to x, we obtain $\pi_2'(x) = \pi_1'(kx)k$. Thus, $-\frac{\pi_2'(x)x}{\pi_2(x)} = -\frac{\pi_1'(kx)kx}{\pi_1(kx)}$. Since $k > 1$ and $-\pi_1'(x)x/\pi_1(x)$ is a strictly decreasing function of x, $-\frac{\pi_1'(kx)kx}{\pi_1(kx)} < -\frac{\pi_1'(x)x}{\pi_1(x)}$. Thus, $-\pi_2'(x)x/\pi_2(x) < -\pi_1'(x)x/\pi_1(x)$, that is, $-\pi_2'(x)/\pi_2(x) < -\pi_1'(x)/\pi_1(x)$ for every x. This is equivalent to $(\pi_2/\pi_1)' > 0$, thus completing the proof. □

We now proceed to prove Propositions 7 and 8. It is convenient for this proof to proceed to a change of variable, as it were. Recall, that $\bar{X}$ is log-normally distributed. Let $s \equiv \log(x)$ for a generic element $x \in \mathbb{R}$ and $S = \log(\bar{X}(\Omega)) = \mathbb{R}$. s is thus normally distributed. Recall the Ω is identified with $\bar{X}(\Omega) \times \mathcal{P}$, which is, in turn, identified with $\mathbb{R} \times \mathbb{R}$. Thus, $\mathbf{P}$ is a joint distribution over $(s, m) \in \mathbb{R} \times \mathbb{R}$. Denote the probability density functions of the second-order belief $\mathcal{N}(\hat{m}, \hat{\sigma}^2)$ and a first-order belief $\mathcal{N}(m, \sigma^2)$ by p_M and $p_{S|M}(\cdot \mid m)$, respectively. It follows from Bayes' formula that the conditional second-order belief given s is

$$\mathcal{N}\left(\frac{\hat{\sigma}^2 s + \sigma^2 \hat{m}}{\sigma^2 + \hat{\sigma}^2}, \frac{\hat{\sigma}^2 \sigma^2}{\sigma^2 + \hat{\sigma}^2}\right). \tag{25}$$

Denote its probability density function by $p_{M|S}(\cdot \mid s)$. Observe we may write the kernel (13) as

$$\pi_{u,\phi}(s) = u'(\exp(s))h(s, \mu), \tag{26}$$

to identify the component h which encapsulates the effect of ambiguity aversion,

$$h(s, \mu) \equiv \int_{\mathcal{P}} \frac{p(s)}{p^{\mu}(s)} \phi'\left(E^{\mathbf{P}} u\left(\bar{X}^{\mathbf{P}}\right)\right) \mu(d\mathbf{P}). \tag{27}$$

Thus, in the case of interest here, (27) can be rewritten as

$$h(s, \hat{m}) = \int \frac{v'(c(m))}{u'(c(m))} p_{M|S}(m \mid s) dm, \tag{28}$$

where $c(m) = u^{-1}\left(E^m u(\bar{X})\right)$ and E^m is the expectation under $\mathcal{N}(m, \sigma^2)$. The relation (26) can be rewritten as $\pi_{u,\phi}(s, \hat{m}) = \lambda(\hat{m}) u'(\exp X(s)) h(s, \hat{m})$. Write $r = \hat{\sigma}^2 / (\sigma^2 + \hat{\sigma}^2)$, then $0 < r < 1$. Denote by q the probability density function of

$$\mathcal{N}\left(0, \frac{\hat{\sigma}^2 \sigma^2}{\sigma^2 + \hat{\sigma}^2}\right).$$

Then, the probability density function of (25) coincides with the function

$s \mapsto q(m - (rs + (1-r)\hat{\sigma}))$ and (28) can be rewritten as

$$h(s, \hat{m}) = \int_{-\infty}^{\infty} \frac{v'(c(m))}{u'(c(m))} q(m - (rs + (1-r)\hat{m}))\, dm.$$

The following two lemmas are consequences of Proposition 13 in Appendix F, which is a general result on strict log-supermodularity (SLSPM for short). They are used to prove Propositions 8 and 7.

Lemma 3. *Suppose that Assumption 1 holds, that u exhibits CRRA, and that the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x. Then, h is strictly log-supermodular, that is,*

$$h(s_1, \hat{m}_1)h(s_2, \hat{m}_2) < h(\max\{s_1, s_2\}, \max\{\hat{m}_1, \hat{m}_2\})h(\min\{s_1, s_2\}, \min\{\hat{m}_1, \hat{m}_2\})$$

for all $(s_1, \hat{m}_1)$ and $(s_2, \hat{m}_2)$, unless $(s_1, \hat{m}_1) \leq (s_2, \hat{m}_2)$ or $(s_1, \hat{m}_1) \geq (s_2, \hat{m}_2)$.

Proof of Lemma 3 By part 1 of Assumption 1,

$$c(m) = \exp\left(m + \frac{\sigma^2}{2}(1-\alpha)\right).$$

Define $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{++}$ by

$$f(s, \hat{m}, m) = \frac{v'(c(m + rs + (1-r)\hat{m}))}{u'(c(m + rs + (1-r)\hat{m}))} q(m).$$

Since $c'(m + rs) = c(m + rs)$,

$$\begin{aligned}
\frac{\partial}{\partial s} \ln f(s, \hat{m}, m) &= \frac{\mathrm{d}}{\mathrm{d}s} \ln v'(c(m + rs + (1-r)\hat{m}))) - \frac{\mathrm{d}}{\mathrm{d}s} \ln u'(c(m + rs + (1-r)\hat{m}))) \\
&= \frac{v''(c(m + rs + (1-r)\hat{m})))}{v'(c(m + rs + (1-r)\hat{m})))} c'(m + rs + (1-r)\hat{m}))r - \\
&\quad - \frac{u''(c(m + rs + (1-r)\hat{m})))}{u'(c(m + rs + (1-r)\hat{m})))} c'(m + rs + (1-r)\hat{m}))r \\
&= \left(\frac{v''(c(m + rs + (1-r)\hat{m})))c(m + rs + (1-r)\hat{m}))}{v'(c(m + rs + (1-r)\hat{m})))} - \alpha\right) r. \qquad (29)
\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial \hat{m}} \ln f(s,\hat{m},m) = \left(\frac{v''(c(m+rs+(1-r)\hat{m}))c(m+rs+(1-r)\hat{m}))}{v'(c(m+rs+(1-r)\hat{m}))} - \alpha\right)(1-r).$$

Thus,

$$\frac{\partial^2}{\partial s \partial \hat{m}} \ln f(s,\hat{m},m) = \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})r(1-r) > 0,$$

since v has differentiably strictly decreasing relative risk aversion. Similarly,

$$\frac{\partial^2}{\partial s \partial m} \ln f(s,\hat{m},m) = \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})r > 0,$$
$$\frac{\partial^2}{\partial \hat{m} \partial m} \ln f(s,\hat{m},m) = \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m+rs+(1-r)\hat{m})(1-r) > 0.$$

Thus, by Proposition 13, the function $(s,\hat{m}) \mapsto \int_{-\infty}^{\infty} f(s,\hat{m},m)\,\mathrm{d}m$ has SLSPM. By the change of variable,

$$\int_{-\infty}^{\infty} f(s,\hat{m},m)\,\mathrm{d}m = \int_{-\infty}^{\infty} \frac{v'(c(m))}{u'(c(m))} b(m-(rs+(1-r)\hat{m}))\,\mathrm{d}m = h(s,\hat{m}). \tag{30}$$

This completes the proof. □

Lemma 4. *Suppose that Assumption 1 holds, that u exhibits CRRA, and that the derivative of $-v''(x)x/v'(x)$ is strictly negative at every x. Then, for every $\hat{m} \in \mathbb{R}$,*

$$\frac{\frac{\partial h}{\partial s}(s,\hat{m})}{h(s,\hat{m})}$$

is strictly increasing in $s \in \mathbb{R}$.

Proof of Lemma 4 Let $\hat{m} \in \mathbb{R}$. Let f be as in the proof of Lemma 3.

Define $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{++}$ by $k(s, \varepsilon, m) = f(s + \varepsilon, \hat{m}, m)$. By (29),

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial \varepsilon} \ln k(s, \varepsilon, m) &= \frac{\partial^2}{\partial s^2} \ln f(s, \hat{m}, m) \\
&= \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m})r^2 > 0, \\
\frac{\partial^2}{\partial m \partial \varepsilon} \ln k(s, \varepsilon, m) &= \frac{\partial^2}{\partial m \partial s} \ln k(s, \varepsilon, m) = \frac{\partial^2}{\partial s \partial m} \ln f(s, \hat{m}, m) \\
&= \frac{\mathrm{d}}{\mathrm{d}y} \frac{v''(y)y}{v'(y)}\bigg|_{y=c(m+rs+(1-r)\hat{m})} c'(m + rs + (1-r)\hat{m})r > 0.
\end{aligned}$$

By Proposition 13, the function $(s, \varepsilon) \mapsto \int_{-\infty}^{\infty} k(s, \varepsilon, m)\,\mathrm{d}m$ has SLSPM. Since $k(s, \varepsilon, m) = f(s + \varepsilon, \hat{m}, m)$, by (30), this function is equal to $(s, \varepsilon) \mapsto h(s + \varepsilon, \hat{m})$. Since it has SLSPM, if $s_1 < s_2$ and $\varepsilon > 0$, then

$$\frac{h(s_1 + \varepsilon, \hat{m})}{h(s_1, \hat{m})} < \frac{h(s_2 + \varepsilon, \hat{m})}{h(_2, \hat{m})}.$$

This means that $h(s + \varepsilon, \hat{m})/h(s, \hat{m})$ is a strictly increasing function of s. Since

$$\frac{d}{ds} \ln \frac{h(s + \varepsilon, \hat{m})}{h(s, \hat{m})} = \frac{\frac{\partial h}{\partial s}(s + \varepsilon, \hat{m})}{h(s + \varepsilon, \hat{m})} - \frac{\frac{\partial h}{\partial s}(s, \hat{m})}{h(s, \hat{m})},$$

and the left-hand side is nonnegative, $\frac{\partial h}{\partial s}(s, \hat{m})/h(s, \hat{m})$ is non-decreasing in s. To prove that it is, in fact, strictly increasing, suppose not. Then, there is an interval, say $(\underline{s}, \overline{s})$, over which it is constant. Take a small $\varepsilon > 0$. Then, over an interval of s with $\underline{s} < s < s + \varepsilon < \overline{s}$, the right-hand side is constantly equal to 0. Hence, $h(s+\varepsilon, \hat{m})/h(s, \hat{m})$ is constant. But this is a contradiction. Thus, $\frac{\partial h}{\partial s}(s, \hat{m})/h(s, \hat{m})$ is strictly increasing in s. □

Proof of Proposition 7 1. This follows from direct calculation.
2. By differentiating the logarithm of (26) and multiplying by -1, we obtain

$$-\frac{\pi'_{u,\phi}(s)}{\pi_{u,\phi}(s)} = -\frac{u''(\exp(s))\exp(s)}{u'(\exp(s))} - \frac{\frac{\partial h}{\partial s}(s, \mu)}{h(s, \mu)}$$

for every s.

$$\varepsilon(x;\pi_{u,\phi}) = -\frac{\pi'_{u,\phi}(x)x}{\pi_{u,\phi}(x)} = -\frac{u''(x)x}{u'(x)} - \frac{\frac{\partial h}{\partial s}(\ln x,\mu)}{h(\ln x,\mu)} \tag{31}$$

for every $x > 0$. Since u exhibits constant relative risk aversion, the first fraction on the right-hand side of (31) (where μ is replaced by $\hat{m}$) is independent of x. By Lemma 4, the second fraction is strictly increasing in x. Thus, $\varepsilon(x;\pi_{u,\phi})$ is strictly decreasing in x. □

Proof of Proposition 8 1. This follows from direct calculation.
2. Let $\hat{m}_2 > \hat{m}_1$. By Lemma 3, $h(s,\hat{m}_2)/h(s,\hat{m}_1)$ is strictly increasing in s. Thus, by (26), where μ is replaced by $\hat{m}_1$ and $\hat{m}_2$, $\pi_{u,\phi}(x;\hat{m}_2)/\pi_{u,\phi}(x;\hat{m}_1)$ is strictly increasing in x. Thus, by part 1 of Lemma 2,

$$\frac{\sigma^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_1)\right)}{E^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_1)\right)} > \frac{\sigma^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}{E^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}. \tag{32}$$

For each n, under the second-order belief $\mathcal{N}(\hat{m}_n,\hat{\sigma}^2)$, the reduced probability over S coincides with $\mathcal{N}(\hat{m}_n,\sigma^2+\hat{\sigma}^2)$. Since $\bar{X}(s) = \exp(s)$, the reduced probability over consumption levels coincides with the log-normal distribution $\mathcal{LN}(\hat{m}_n,\hat{\sigma}^2+\sigma^2)$. Let g_n be the probability density function of this distribution and $k = \exp(\hat{m}_2 - \hat{m}_1)$, then $k > 1$ and $g_1(x) = kg_2(kx)$ for every $x > 0$. Thus, by part 2 of Lemma 2,

$$\frac{\sigma^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}{E^{\hat{m}_1}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)} > \frac{\sigma^{\hat{m}_2}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}{E^{\hat{m}_2}\left(\pi_{u,\phi}(\,\cdot\,;\hat{m}_2)\right)}. \tag{33}$$

By (32) and (33), the proof is completed. □

Proposition 9. *For each $n = 1,2$, let $\pi_n : \mathbb{R}_{++} \to \mathbb{R}_{++}$ be differentiable and suppose that $\pi'_n < 0$. Suppose, moreover, that $\varepsilon(s;\pi_1)$ is independent of s, $\varepsilon(s;\pi_2)$ is strictly decreasing in s, and the value of the former is contained in the range of the latter. Suppose, furthermore, that there is a non-degenerate probability P on $\mathbb{R}_{++}$ s.th. $\int \pi_1(x)P(dx) = \int \pi_2(x)P(dx)$. Then, there are*

x_* and x^* in $\mathbb{R}_{++}$ with $x_* < x^*$ s.th. $\pi_1(x) < \pi_2(x)$ if $x < x_*$ or $x > x^*$; $\pi_1(x) > \pi_2(x)$ if $x_* < x < x^*$; and $\pi_1(x) = \pi_2(x)$ if $x = x_*$ or $x = x^*$.

Proof of Proposition 9 Define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = \ln \pi_2(\exp z) - \ln \pi_1(\exp z)$. Then,

$$g'(z) = \frac{\pi_2'(\exp z) \exp z}{\pi_2(\exp z)} - \frac{\pi_1'(\exp z) \exp z}{\pi_1(\exp z)}.$$

Thus, g' is strictly increasing, and there are $\underline{z}$ and $\overline{z}$ s.th. $g'(\underline{z}) < 0 < g'(\overline{z})$. Then, $g'(z) \leq g'(\underline{z})$ for every $z \leq \underline{z}$ and $g'(z) \geq g'(\overline{z})$ for every $z \geq \overline{z}$. By applying the mean-value theorem to g on the interval $[z, \underline{z}]$ and the strict increasingness of g', we obtain $g(\underline{z}) \leq g'(\underline{z})(\underline{z} - z) + g(z)$, that is, $g(z) \geq -g'(\underline{z})(\underline{z} - z) + g(\underline{z})$ for every $z < \underline{z}$. As $z \to -\infty$, the right-hand side diverges to ∞. Similarly, $g(z) \geq g'(\overline{z})(z - \overline{z}) + g(\overline{z})$ for every $z > \overline{z}$. As $z \to \infty$, the right-hand side diverges to ∞. Thus, g attains its minimum (over the entire $\mathbb{R}$). Denote by $\hat{z}$ a point at which the minimum is attained. Then, $g'(\hat{z}) = 0$ by the first-order condition. Since g' is strictly increasing, $g'(z) < 0$ for every $z < \hat{z}$, and $g'(z) > 0$ for every $z > \hat{z}$. Thus, g is strictly decreasing on $(-\infty, \hat{z})$ and strictly increasing on $(\hat{z}, -\infty)$.

If $g(\hat{z}) \geq 0$, then $g(z) \geq 0$ for every z, with a strict inequality possibly except at $z = \hat{z}$. Thus, $\pi_2(x) \geq \pi_1(x)$ for every x, with a strict inequality possibly except for $x = \exp \hat{z}$, and the integral assumption is violated. Thus, $g(\hat{z}) < 0$. By the intermediate value theorem, there is a unique $z_* < \hat{z}$ s.th. $g(z_*) = 0$; and there is a unique $z^* > \hat{z}$ s.th. $g(z^*) = 0$. Let $x_* = \exp z_*$ and $x^* = \exp z^*$, to complete the proof. □

Parameters for Figure 2

x is lognormal with mean and volatility parameters m and σ, that are unknown. Consumers put probability 1/2 on (m_1, σ_1) and 1/2 on (m_2, σ_2). For the parameter values, we rely on the two-regime (annualized) specification in Table 6 of (Gadea, Gómez-Loscos & Pérez-Quirós 2020). (Gadea, Gómez-Loscos & Pérez-Quirós 2020) divided the time span of data (from 1875 to 2014) into two historical regimes. We assume a recession partially identifies the distributions as a set of two possible distributions because consumers think the recessionary distributions in either historical regime is possible. Analogously, an expansion also partially identifies a set of two distributions. Following the argument in Section 2.3, consumers behave as if the worst distribution in each partially-identified set is in operation. Thus, we obtain $(m_1, \sigma_1) = (.04, .011)$ ($\mathbf{P}_B$), and $(m_2, \sigma_2) = (-.15, .11)$ ($\mathbf{P}_b$). The four economies considered are:

Homogeneous and ambiguity neutral : EU representative consumer with a CRRA utility function with relative risk aversion equal to 2/3.

Homogeneous and ambiguity averse : smooth ambiguity representative consumer with CRRA u with relative risk aversion equal to 2/3 and CRRA v with index 12.

Homogeneous and ambiguity averse : smooth ambiguity representative consumer with CRRA u with relative risk aversion equal to 2/3 and CRRA v with index 6.

Heterogeneous and ambiguity averse There is a consumer of each of the two ambiguity averse types described above, with equal weight.

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Online-Appendix

A Ambiguity aversion and revealed beliefs

Denote by $b(\Omega_{\mathbf{P}})$ a bet that pays c^* on $\Omega_{\mathbf{P}}$ and c_* off it, and by $b(\Omega \setminus \Omega_{\mathbf{P}})$ the bet on the complementary event $\Omega \setminus \Omega_{\mathbf{P}}$. Note that $\mathbf{P}(\Omega_{\mathbf{P}}) = 1$. Normalize u_i and ϕ_i so that $u_i(c_*) = 0$ and $\phi_i(0) = 0$, and write $h = u(c^*)$. Consumer i evaluates these bets as: $U_i(b(\Omega_{\mathbf{P}})) = \mu(\mathbf{P})\,\phi_i(\mathbf{P}(\Omega_{\mathbf{P}})\,h) = \mu(\mathbf{P})\,\phi_i(h)$ and $U_i(b(\Omega \setminus \Omega_{\mathbf{P}})) = (1-\mu(\mathbf{P}))\,\phi_i(h)$. Consider next a lottery ℓ^π which pays c^* with a probability π and c_* with probability $1-\pi$. Then, $U_i(\ell^\pi) = \phi_i(\pi h)$. If ϕ_i is strictly concave, then $U_i(b(\Omega_{\mathbf{P}})) < U_i(\ell^{\mu(\mathbf{P})})$ and $U_i(b(\Omega \setminus \Omega_{\mathbf{P}})) < U_i(\ell^{1-\mu(\mathbf{P})})$. Define $\underline{\pi}, \bar{\pi} \in [0,1]$ so that $U_i(\ell^{\underline{\pi}}) = U_i(b(\Omega_{\mathbf{P}}))$ and $U_i(\ell^{1-\bar{\pi}}) = U_i(b(\Omega \setminus \Omega_{\mathbf{P}}))$. Since ϕ is strictly increasing, $\underline{\pi} < \mu(\mathbf{P}) < \bar{\pi}$. Moreover, $\underline{\pi}$ satisfies

$$\begin{aligned}
\phi_i(\underline{\pi} h + (1-\underline{\pi})0) &= \mu(\mathbf{P})\,\phi_i(h) + (1-\mu(\mathbf{P}))\phi_i(0) \\
\Leftrightarrow \underline{\pi} h &= \phi_i^{-1}(\mu(\mathbf{P})\,\phi_i(h))
\end{aligned}$$

Applying a quadratic approximation, we get, letting λ_{ϕ_i} be the Arrow-Pratt measure of absolute risk aversion for the function ϕ_i (see Online-Appendix B for further detail).

$$\begin{aligned}
\underline{\pi} h &= \mu(\mathbf{P})\,h - \frac{\lambda_{\phi_i}(0)}{2}\left[\mu(\mathbf{P})\,h^2 - (\mu(\mathbf{P})\,h)^2\right] + o\left(h^2\right) \\
\Leftrightarrow \underline{\pi} &= \mu(\mathbf{P}) - \frac{\lambda_{\phi_i}(0)}{2}\mu(\mathbf{P})\,(1-\mu(\mathbf{P}))\,h + o(h)
\end{aligned}$$

Similarly, $\bar{\pi} = \mu(\mathbf{P}) + \frac{\lambda_{\phi_i}(0)}{2}\mu(\mathbf{P})\,(1-\mu(\mathbf{P}))\,h + o(h)$. Hence, the "probability matching" interval for $\Omega_{\mathbf{P}}$ is given by $[\underline{\pi}, \bar{\pi}]$. Its length is increasing in λ_{ϕ_i}.

B Relative ambiguity aversion

We relate the measure of relative ambiguity aversion introduced in Section 2.2 to ambiguity premiums (see also (Cerreia-Vioglio, Maccheroni & Marinacci 2022)). Let h be a random variable defined on Ω and w be the initial consumption level. Denote by $\mathbf{P}^\mu$ the reduced measure $\int_{\mathcal{P}} \mathbf{Q}\mu(d\mathbf{Q})$, and by λ_u the Arrow-Pratt measure of absolute risk aversion for a Bernoulli utility u. The variance $(\sigma^{\mu})^2 (E^{\cdot}(h))$ of the function $E^{\cdot}(h) : \mathbf{P} \mapsto E^{\mathbf{P}}(h)$ under μ reflects the uncertainty on the expected values and encapsulates ambiguity. The certainty equivalent for a proportional ambiguous prospect xh can be approximated as[31]

$$\begin{aligned} C(x+xh) &= x + E^{\mathbf{P}^{\mu}}(xh) - \frac{x^2}{2}\lambda_u(x)(\sigma^{\mathbf{P}^{\mu}})^2(h) \\ &\quad -\frac{x^2}{2}(\lambda_v(x)-\lambda_u(x))(\sigma^{\mu})^2\left(E^{\mathbf{P}^{\mu}}(h)\right) + o\left(\|h\|^2\right) \end{aligned}$$

Since $\phi = v \circ u^{-1}$, $\lambda_\phi(u(x)) = \frac{1}{u'(x)}(\lambda_v(x) - \lambda_u(x))$, that is, $\lambda_\phi(u(x))u'(x) = \lambda_v(x) - \lambda_u(x)$. The ambiguity premium for xh is obtained by subtracting the risk premium from the overall uncertainty premium and, as a proportion of wealth, equal to

$$((\lambda_v(x)-\lambda_u(x))x) \times \frac{1}{2}(\sigma^{\mu})^2\left(E^{\mathbf{P}^{\mu}}(h)\right) = \lambda_\phi(u(x))u'(x)\,x \times \frac{1}{2}(\sigma^{\mu})^2\left(E^{\mathbf{P}^{\mu}}(h)\right).$$

In our HARA specification, it is convenient to express the ambiguity premium in terms of the effective consumption $x-\zeta$. By differentiating $v = \phi \circ u$, we obtain

$$-\frac{v''(x)}{v'(x)} = -\frac{\phi''(u(x))}{\phi'(u(x))}u'(x) - \frac{u''(x)}{u'(x)}. \tag{34}$$

By multiplying both sides by $x-\zeta$, we obtain, under Condition 2:

$$-\frac{\phi''(u(x))}{\phi'(u(x))}u'(x)(x-\zeta) = \gamma - \alpha. \tag{35}$$

C Proof of Proposition 5.

For the purpose of this Appendix, denote the value function of (8) by $u(x,\lambda)$. Then, u is the representative consumer's (inner) Bernoulli utility function, where dependence on the vector λ of utility weights is made explicit. Similarly, denote the value function of (4) by $V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)$ and the value function of (2) by $V\left((\bar{X}^{\mathbf{P}})_{\mathbf{P}\in\mathcal{P}},\lambda\right)$. Then,

$$V\left((\bar{Y}^{\mathbf{P}})_{\mathbf{P}},\lambda\right) = \sum_{\mathbf{P}} \mu(\mathbf{P}) V^{P}\left(\bar{Y}^{\mathbf{P}},\lambda\right) \tag{36}$$

[31]This is akin to the quadratic approximation of certainty equivalent obtained by (Maccheroni, Marinacci & Ruffino 2013)

for every $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$.

Denote the solution to (8) by $(f_i(x,\lambda))_i$. Then $(f_i)_i$ is the risk-sharing rule, with the dependence on the vector λ of utility weights made explicit. By the envelope theorem,

$$\frac{\partial u}{\partial x}(x,\lambda) = \lambda_i u_i'(f_i(x,\lambda)) \tag{37}$$

for every i. Denote the risk tolerances of u_i and u by t_i and t. (Wilson 1968) showed that $t(x,\lambda) = \sum_i t_i(f_i(x,\lambda))$ for every (x,λ). Hence,

$$\nabla_\lambda t(x,\lambda) = \sum_i t_i'(f_i(x,\lambda)) \nabla_\lambda f_i(x,\lambda). \tag{38}$$

Lemma 5. $\nabla_\lambda t(x,\lambda) = 0$ *if and only if* $t_1'(f_1(x,\lambda)) = \cdots = t_I'(f_I(x,\lambda))$.

Proof of Lemma 5 Although this lemma is true for an arbitrary I, we give a proof only for $I=2$ to save space. By (37), $\lambda_1 u_1'(f_1(x,\lambda)) = \lambda_2 u_2'(f_2(x,\lambda))$. By differentiating both sides w.r.t. λ_1:

$$u_1'(f_1(x,\lambda)) + \lambda_1 u_1''(f_1(x,\lambda))\frac{\partial f_1}{\partial \lambda_1}(x,\lambda) = \lambda_2 u_2''(f_2(x,\lambda))\frac{\partial f_2}{\partial \lambda_1}(x,\lambda).$$

Since $\sum_i (\partial f_i/\partial \lambda_1)(x,\lambda) = 0$, $u_1'(f_1(x,\lambda)) = -\frac{\partial f_1}{\partial \lambda_1}(x,\lambda)\sum_i \lambda_i u_i''(f_i(x,\lambda))$. Hence, $(\partial f_1/\partial\lambda_1)(x,\lambda) > 0$. Thus,

$$\frac{\partial t}{\partial \lambda_1}(x,\lambda) = \left(t_1'(f_1(x,\lambda)) - t_2'(f_2(x,\lambda))\right)\frac{\partial f_1}{\partial \lambda_1}(x,\lambda) = 0$$

if and only if $t_1'(f_1(x,\lambda)) = t_2'(f_2(x,\lambda))$. □

If $(X_i^{\mathbf{P}})_i$ is a solution to (4), then, by the envelope theorem,

$$\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}(\omega)} = \lambda_i \phi_i'\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right) u_i'(X_i^{\mathbf{P}}(\omega))\mathbf{P}(\omega) \text{ for all } i \text{ and } \omega. \tag{39}$$

Lemma 6. *For each* $\mathbf{P} \in \mathcal{P}$*, let* $(X_i^{\mathbf{P}})_i$ *be a solution to* (4)*. Write* $\lambda_i^{\mathbf{P}} = \lambda_i \phi_i'\left(E^{\mathbf{P}} u_i(X_i^{\mathbf{P}})\right)$ *and* $\lambda^{\mathbf{P}} = (\lambda_i^{\mathbf{P}})_i$*. Suppose that there is a pair of a differentiable function* $u : \mathbb{X} \to \mathbb{R}$ *and a differentiable function* $\phi : u(\mathbb{X}) \to \mathbb{R}$ *such that* $V^{\mathbf{P}}(\bar{Y},\lambda) = \phi(E^{\mathbf{P}} u(\bar{Y}))$ *for all* $\mathbf{P} \in \mathcal{P}$ *and* $\bar{Y} : \Omega \to \mathbb{X}$*. Then, for all* ω_1 *and* $\omega_2 \in \Omega_{\mathbf{P}}$*,*

$$\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})}$$

depends only on the values of $\bar{X}^{\mathbf{P}}(\omega_1)$ *and* $\bar{X}^{\mathbf{P}}(\omega_2)$*, that is, if* $\bar{X}^{\mathbf{P}}(\omega_1) = \bar{X}^{\mathbf{Q}}(\omega_3)$ *and* $\bar{X}^{\mathbf{P}}(\omega_2) = \bar{X}^{\mathbf{Q}}(\omega_4)$*, then* $\int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})} = \int_{\bar{X}^{\mathbf{Q}}(\omega_3)}^{\bar{X}^{\mathbf{Q}}(\omega_4)} \frac{dx}{t(x,\lambda^{\mathbf{Q}})}$ *for all* $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$*,* $\omega_1, \omega_2 \in \Omega_{\mathbf{P}}$*, and* $\omega_3, \omega_4 \in \Omega_{\mathbf{Q}}$*.*

Proof of Lemma 6 First, we prove that

$$\frac{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \exp\left(- \int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})} \right).$$

Indeed, by (39),

$$\frac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega)}}{\mathbf{P}(\omega)} = \lambda_i^{\mathbf{P}} u_i'(X_i^{\mathbf{P}}(\omega))$$

for every ω. Thus, the right-hand side is independent of i. Hence, the first-order condition for a solution to (8) is met, and $X_i^{\mathbf{P}}(\omega) = f_i(\bar{X}^{\mathbf{P}}(\omega), \lambda^{\mathbf{P}})$ for all i and $\omega \in \Omega$. Thus, by (37), $\lambda_i^{\mathbf{P}} u_i'(X_i^{\mathbf{P}}(\omega)) = \frac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega), \lambda^{\mathbf{P}})$. Hence,

$$\frac{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\dfrac{\dfrac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \frac{\dfrac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega_2), \lambda^{\mathbf{P}})}{\dfrac{\partial u}{\partial x}(\bar{X}^{\mathbf{P}}(\omega_1), \lambda^{\mathbf{P}})} = \exp\left(- \int_{\bar{X}^{\mathbf{P}}(\omega_1)}^{\bar{X}^{\mathbf{P}}(\omega_2)} \frac{dx}{t(x,\lambda^{\mathbf{P}})} \right). \qquad (40)$$

On the other hand, by assumption, the chain rule implies that

$$\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega)} = \phi'(E^{\mathbf{P}} u(\bar{X}))u'(\bar{X}^{\mathbf{P}}(\omega))\mathbf{P}(\omega)$$

for every ω. Thus,

$$\frac{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_2)}}{\mathbf{P}(\omega_2)}}{\frac{\frac{\partial V^{\mathbf{P}}(\bar{X}^{\mathbf{P}},\lambda)}{\partial \bar{X}^{\mathbf{P}}(\omega_1)}}{\mathbf{P}(\omega_1)}} = \frac{u'(\bar{X}^{\mathbf{P}}(\omega_2))}{u'(\bar{X}^{\mathbf{P}}(\omega_1))}$$

for all ω_1 and ω_2. Since the right-hand side depends only on the values of $\bar{X}^{\mathbf{P}}(\omega_1)$ and $\bar{X}^{\mathbf{P}}(\omega_2)$, so is the left-hand side. The lemma follows now from (40). □

Lemma 7. *Suppose that there is a pair of a differentiable function $u : \mathbb{X} \to \mathbb{R}$ and a differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ such that $V((\bar{Y}^{\mathbf{P}})_{\mathbf{P}}, \lambda) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi(E^{\mathbf{P}}u(\bar{Y}^{\mathbf{P}}))$ for every $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$, where $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$ for every $\mathbf{P}$. Then, $V^{\mathbf{P}}(\bar{Y}, \lambda) = \phi(E^{\mathbf{P}}u(\bar{Y}))$ for all $\mathbf{P}$ and $\bar{Y} : \Omega \to \mathbb{X}$.*

Proof of Lemma 7 Let $\mathbf{Q} \in \mathcal{P}$, and $(\bar{X}^{\mathbf{P}})_{\mathbf{P}}$ and $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$ be two aggregate endowments such that $\bar{X}^{\mathbf{P}} = \bar{Y}^{\mathbf{P}}$ for every $\mathbf{P} \in \mathcal{P} \setminus \{\mathbf{Q}\}$. By assumption and (36),

$$\phi(E^{\mathbf{Q}}u(\bar{X}^{\mathbf{Q}})) - \phi(E^{\mathbf{Q}}u(\bar{Y}^{\mathbf{Q}})) = V^{\mathbf{Q}}\left(\bar{X}^{\mathbf{Q}}, \lambda\right) - V^{\mathbf{Q}}\left(\bar{Y}^{\mathbf{Q}}, \lambda\right).$$

Therefore, for every $\mathbf{P} \in \mathcal{P}$, there is an $a^{\mathbf{P}} \in \mathbb{R}$ such that $V^{\mathbf{P}}(\bar{Y}^{\mathbf{P}}) = \phi(E^{\mathbf{P}}u(\bar{Y}^{\mathbf{P}})) + a^{\mathbf{P}}$ for every $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$. Hence, $\sum_{\mathbf{P}} \mu(\mathbf{P})a^{\mathbf{P}} = 0$. Let $\bar{Y} : \Omega \to \mathbb{X}$ be (deterministic) aggregate endowments for which there is an $x \in \mathbb{X}$ such that $\bar{Y}(\omega) = x$ for every ω. Then, for every $\mathbf{P}$, the solution $(Y_i^{\mathbf{P}})_i$ to (2) is given by letting $Y_i^{\mathbf{P}}$ be the deterministic consumption x_i such that $\lambda_i\phi_i'(u_i(x_i))u_i'(x_i)$ is independent of i, and $V^{\mathbf{P}}(\bar{Y}) = \sum_i \lambda_i\phi_i(u_i(x_i))$. Thus, whenever $\bar{Y}$ is deterministic, $V^{\mathbf{P}}(\bar{Y})$ is independent of $\mathbf{P}$. Hence, $a^{\mathbf{P}}$ is independent of $\mathbf{P}$. Thus, $a^{\mathbf{P}} = 0$ for every $\mathbf{P}$. Hence, $V^{\mathbf{P}}(\bar{Y}) = \phi(E^{\mathbf{P}}u(\bar{Y}))$ for all $\mathbf{P}$ and $\bar{Y}^{\mathbf{P}} : \Omega_{\mathbf{P}} \to \mathbb{X}$. □

Proposition 10. *Assume $|\Omega| \geq 4$. For each i, let u_i be a (inner) Bernoulli utility function with the following property: for each i, there is an $x_i^* \in \mathbb{X}_i$ such that it is not true that $t_1'(x_1^*) = t_2'(x_2^*) = \cdots = t_I'(x_I^*)$. Then, there are: for each i, a Bernoulli utility function ϕ_i over expected utility levels; a* *(common) second-order belief μ on Ω; aggregate endowments $\bar{X}:\Omega\to\mathbb{X}$ whose range is model-independent; and a vector $\lambda^\star$ of utility weights, such that such that if V is defined by (2), then there is no pair of a (inner) Bernoulli utility function u and a Bernoulli utility function ϕ over expected utility levels such that $V\left((\bar{Y}^{\mathbf{P}})_{\mathbf{P}},\lambda\right)=\sum_{\mathbf{P}}\mu(\mathbf{P})\phi\left(E^{\mathbf{P}}u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$.*

Proof of Proposition 10 Suppose that for each i, there is an $x_i^*\in\mathbb{X}_i$ such that it is not true that $t_1'(x_1^*)=t_2'(x_2^*)=\cdots=t_I'(x_I^*)$. For each i, let $\lambda_i^*=(u_i'(x_i^*))^{-1}$, and $\lambda^*=(\lambda_i^*)_i$. Write $x^*=\sum_i x_i^*$. Then, $x_i^*=f_i(x^*,\lambda^*)$ for every i. By Lemma 5, $\nabla_\lambda t(x^*,\lambda^*)$ is a nonzero vector. Thus, there is a $\kappa\in\mathbb{R}^I$ such that $\nabla_\lambda t(x^*,\lambda^*)\kappa>0$. Note here that

$$D_\lambda u_i(f_i(x^*,\lambda^*))=u_i'(f_i(x^*,\lambda^*))\nabla_\lambda f_i(x^*,\lambda^*)\in\mathbb{R}^I.$$

Let $\delta>0$ be so large that $D_\lambda u_i(f_i(x^*,\lambda^*))\kappa+\delta>0$ for every i, then there is a neighborhood $\mathbb{Y}$ of x^* and a neighborhood Λ of λ^* such that $D_\lambda u_i(f_i(x,\lambda))\kappa+\delta>0$ and $\nabla_\lambda t(x,\lambda)\kappa>0$ for all i and $(x,\lambda)\in\mathbb{Y}\times\Lambda$. Then,

$$\frac{d}{d\varepsilon}t(x,\lambda^*+\varepsilon\kappa)=\nabla_\lambda t(x,\lambda^*+\varepsilon\kappa)\kappa>0$$

for every $x\in\mathbb{Y}$ and every ε sufficiently close to 0. Hence, for every $x\in\mathbb{Y}$, $t(x,\lambda^*+\varepsilon\kappa)$ is a strictly increasing function of ε around 0.

Since $\Omega\geq 4$, there is a partition $(\Xi^1,\Xi^2,\Xi^3,\Xi^4)$ of Ω where each Ξ^n is non-empty. Let $x^1,x^2\in\mathbb{X}$ be such that $x^1<x^2$. Define $\bar{X}:\Omega\to\mathbb{X}$ by

$$\bar{X}(\omega)=\begin{cases}x^1 & \text{if } \omega\in\Xi^1\cup\Xi^3\\ x^2 & \text{if } \omega\in\Xi^2\cup\Xi^4\end{cases}$$

Define $\rho>0$ so that $(u_i(f_i(x^2,\lambda))-u_i(f_i(x^1,\lambda)))\rho>\delta$ for all i. Let $\mathbf{P}^0\in\Delta(\Omega)$ be s. th. $\mathbf{P}^0(\omega)>0$ for all $\omega\in\Xi^1\cup\Xi^2$ and $\mathbf{P}^0(\omega)=0$ for all $\omega\in\Xi^3\cup\Xi^4$. For each $\varepsilon>0$ sufficiently close to 0, let $\mathbf{P}^\varepsilon\in\Delta(\Omega)$ be s. th.

$$\mathbf{P}^\varepsilon(\omega)=\begin{cases}\frac{1}{|\Xi^3|}(\mathbf{P}^0(\Xi^1)-\varepsilon\rho) & \text{if } \omega\in\Xi^3,\\ \frac{1}{|\Xi^4|}(\mathbf{P}^0(\Xi^2)+\varepsilon\rho) & \text{if } \omega\in\Xi^4,\\ 0 & \text{if } \omega\in\Xi^1\cup\Xi^2\end{cases}$$

Then, $\mathbf{P}^\varepsilon(\Xi^3)=\mathbf{P}^0(\Xi^1)-\varepsilon\rho$, $\mathbf{P}^\varepsilon(\Xi^4)=\mathbf{P}^0(\Xi^2)+\varepsilon\rho$ and $\mathbf{P}^\varepsilon(\Xi^1\cup\Xi^2)=0$. Fix a sufficiently small $\varepsilon^\star>0$ and let $\mathcal{P}=\{\mathbf{P}^0,\mathbf{P}^{\varepsilon^\star}\}$. Then, $\mathcal{P}$ is point-identified with kernel k s.th.

$$k(\omega)=\begin{cases}\mathbf{P}^0 & \text{if } \omega\in\Xi^1\cup\Xi^2\\ \mathbf{P}^{\varepsilon^\star} & \text{if } \omega\in\Xi^3\cup\Xi^4\end{cases}$$

Moreover, $\Omega_{\mathbf{P}^0} = \Xi^1 \cup \Xi^2$ and $\Omega_{\mathbf{P}^{\varepsilon^\star}} = \Xi^3 \cup \Xi^4$. Thus, the range of $\bar{X}$ is model independent. Let μ be a second-order belief s.th. $\mu(\mathbf{P}^0) > 0$ and $\mu(\mathbf{P}^{\varepsilon^\star}) > 0$. Then, by definition of δ and ρ, $\forall \varepsilon > 0$:

$$\begin{aligned}\frac{d}{d\varepsilon} E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)) &= (u_i(f_i(x^2, \lambda^* + \varepsilon\kappa))) - u_i(f_i(x^1, \lambda^* + \varepsilon\kappa))\rho \\ &\quad + \sum_{\omega\in\Omega} \mathbf{P}^\varepsilon(\omega) D_\lambda u_i(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))\kappa \\ &> \delta + \sum_{\omega\in\Omega} \mathbf{P}^\varepsilon(\omega)(-\delta) = 0.\end{aligned}$$

Thus, by Proposition 10 of (Hara et al. 2022) for each i, there is a twice continuously differentiable ϕ_i with $\phi_i'' \leq 0 < \phi_i'$ such that $\left(\left(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)\right)_i\right)_{\varepsilon=0,\varepsilon^\star}$ is an efficient allocation of the economy $\left((u_i, \phi_i, \mu)_i, \bar{X}\right)$.

Since $\left(\left(f_i(\bar{X}, \lambda^* + \varepsilon\kappa)\right)_i\right)_{\varepsilon=0,\varepsilon^\star}$ is an efficient allocation of the economy $\left((u_i, \phi_i, \mu)_i, \bar{X}\right)$, there is a $\nu \in \mathbb{R}^I_{++}$ such that it is a solution to (2) when λ is replaced by ν. The first-order condition is that for all ε and ω,

$$\nu_i \phi_i'\left(E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa))\right) u_i'(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))$$

is independent of i. Write $\lambda_i^{\mathbf{P}^\varepsilon} = \nu_i \phi_i'\left(E^{\mathbf{P}^\varepsilon} u_i(f_i(\bar{X}, \lambda^* + \varepsilon\kappa))\right)$. By definition,

$$(\lambda_i^* + \varepsilon\kappa_i) u_i'(f_i(\bar{X}(\omega), \lambda^* + \varepsilon\kappa))$$

is independent of i. Thus, $\lambda_i^{\mathbf{P}^\varepsilon}/(\lambda_i^* + \varepsilon\kappa_i)$ is independent of i. Denote it by c^ε. Then $\lambda^{\mathbf{P}^\varepsilon} = c^\varepsilon(\lambda^* + \varepsilon\kappa)$. Hence, $u\left(\cdot, \lambda^{\mathbf{P}^\varepsilon}\right) = c^\varepsilon u\left(\cdot, \lambda^* + \varepsilon\kappa\right)$. Thus, $t\left(\cdot, \lambda^{\mathbf{P}^\varepsilon}\right) = t\left(\cdot, \lambda^* + \varepsilon\kappa\right)$. Hence,

$$\int_{x^1}^{x^2} \frac{dx}{t(x, \lambda^{\mathbf{P}^\varepsilon})} = \int_{x^1}^{x^2} \frac{dx}{t(x, \lambda^* + \varepsilon\kappa)}. \tag{41}$$

Since $t(x, \lambda^* + \varepsilon\kappa)$ is a strictly increasing function of ε for every x, each side of this equality is a strictly decreasing function of ε. In particular, each side is greater for $\varepsilon = 0$ than for $\varepsilon = \varepsilon^\star$.

Suppose that there is a pair of a twice continuously differentiable function $u : \mathbb{X} \to \mathbb{R}$ satisfying $u'' < 0 < u'$ and a twice continuously differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ satisfying $\phi'' \leq 0 < \phi'$ such that $V(\left(\bar{Y}^{\mathbf{P}}\right)_{\mathbf{P}}) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$, where V is the value function of (2). Then, by Lemma 7, for every $\mathbf{P} \in \mathcal{P}$, $V^{\mathbf{P}}(\bar{Y}^{\mathbf{P}}) = \phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $\bar{Y}^{\mathbf{P}}$:

$\Omega \to \mathbb{X}$, where $V^{\mathbf{P}}$ is the value function of (4). Thus, by Lemma 6, the left-hand side of (41) is independent of ε. In particular, it takes the same value for $\varepsilon = 0$ and $\varepsilon = \varepsilon^\star$. This is a contradiction. Hence, there is no pair of a differentiable function $u : \mathbb{X} \to \mathbb{R}$ and a differentiable function $\phi : u(\mathbb{X}) \to \mathbb{R}$ such that $V(\left(\bar{Y}^{\mathbf{P}}\right)_{\mathbf{P}}) = \sum_{\mathbf{P}} \mu(\mathbf{P})\phi\left(E^{\mathbf{P}} u(\bar{Y}^{\mathbf{P}})\right)$ for all $(\bar{Y}^{\mathbf{P}})_{\mathbf{P}}$. □

D Constant absolute risk aversion

We study here an economy where u_i and v_i are HARA with zero marginal risk tolerance.[32]

Assumption 2. *Assume u_i is CARA with risk aversion $\alpha_i > 0$ and v_i is CARA with risk aversion $\gamma_i \geq \alpha_i$*

Assumption 2 is equivalent to assume u_i and v_i are HARA with CMRT (with parameters $(0, \frac{1}{\alpha_i})$ and $(0, \frac{1}{\gamma_i})$ respectively). Let $\phi_i = v_i \circ u_i^{-1}$, so $\phi_i(t) \propto -(-t^{\gamma_i/\alpha_i})$. Hence, our economy consists of smooth ambiguity averse consumers with heterogeneous risk aversion and heterogeneous ambiguity aversion, parameterized by CARA Bernoulli utilities with risk aversion coefficient $\alpha_i > 0$ and by a power function with index $\frac{\gamma_i}{\alpha_i} \geq 1$, respectively.

Proposition 11. *Let $(X_i^{\mathbf{P}})_{\mathbf{P},i}$ be an efficient allocation of an economy that satisfies Assumption 2. Let $\alpha = \left(\sum_i \alpha_i^{-1}\right)^{-1}$ and $\gamma = \left(\sum_i \gamma_i^{-1}\right)^{-1}$. Then,*

1. *For each P, there are constants $(\tau_i^{\mathbf{P}})_{i=1,\dots,I}$ s.th. $\sum_i \tau_i^{\mathbf{P}} = 0$ and $X_i^{\mathbf{P}} = (\alpha/\alpha_i)\bar{X} + \tau_i^{\mathbf{P}}$ for every i.*

2. *For every i, there is a function $\tau_i : (-\infty, \infty) \to (-\infty, \infty)$ and constants κ_i such that $\tau_i(c) = \frac{\gamma}{\gamma_i}\left(1 - \frac{\gamma_i/\alpha_i}{\gamma/\alpha}\right) c + \kappa_i$ with $\sum_i \kappa_i = 0$ and*
$$\tau_i^{\mathbf{P}} = \tau_i(c^{\mathbf{P}}) \tag{42}$$
   with $c^{\mathbf{P}} = u^{-1}(E^{\mathbf{P}} u(\bar{X}))$, where u, the representative consumer's utility function, is CARA with absolute risk aversion coefficient α.

3. *In the smooth ambiguity representative consumer's utility $\phi(t) \propto -(-t^{\gamma/\alpha})$ and $v = \phi \circ u$ is CARA with parameter γ.*

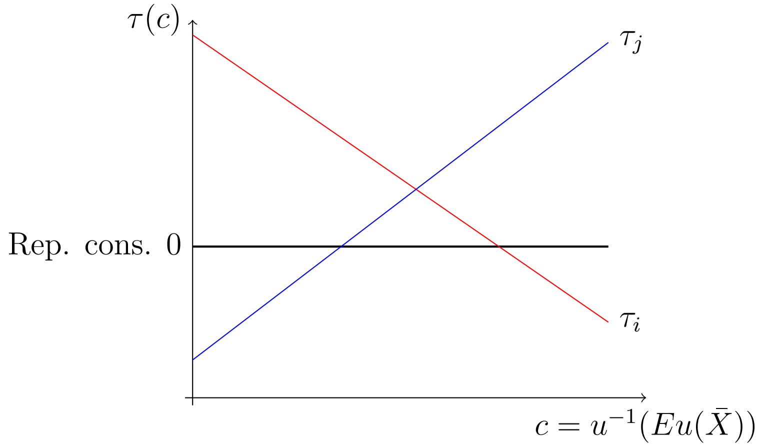

Figure 3: Constant risk tolerance case. The Figure shows the transfers as a function of the certainty equivalents for two consumers, i and j. Consumer i is more ambiguity averse than, and j is less ambiguity averse than, the representative consumer.

As P varies, the efficient allocation rule adjusts by varying the intercept term of the linear sharing rule, $\tau_i^{\mathbf{P}}$, a term denoting transfers that sum to zero across all the consumers. The function $\tau_i^{\mathbf{P}}$ is itself linear in the aggregate certainty equivalent. Figure (3) gives a graphical depiction showing how $\tau_i^{\mathbf{P}}$ varies as a function of the representative consumer's certainty equivalent for two consumers in this economy as established in Proposition 11.

If ambiguity attitudes were homogeneous, i.e., $\gamma_i/\alpha_i = \gamma_j/\alpha_j$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected utility consumers: for all i, $\tau_i^{\mathbf{P}}$ is independent of $\mathbf{P}$.

E Non-zero marginal risk tolerance

We provide here a complement to Proposition 6 and give the limit behavior of $\theta_i(.)$ and b.

Proposition 12. *Consider the functions θ_i and RAA_ϕ constructed in Proposition 6. Then,*

[32]While this class of utility functions is usually not the one considered in the DSGE literature, it admits an easy representation for the efficient allocations and the representative consumer's utility function, while allowing for heterogeneity.

1. *$\theta_i(z) \to 0$ as $z \to 0$ if $\gamma_i \neq \max_{i=1,\dots,I} \gamma_i$ and $\theta_i(z) \to 0$ as $z \to \infty$ if $\gamma_i \neq \min_{i=1,\dots,I} \gamma_i$.*

2. *$RAA_\phi(z) \to \max_{i=1,\dots,I} \gamma_i - \alpha$ as $z \to 0$, and $RAA_\phi(z) \to \min_{i=1,\dots,I} \gamma_i - \alpha$ as $z \to \infty$.*

Proof of Proposition 12 The l.h.s. of (21) is equal to the derivative of the logarithm of the function $z \mapsto (f_i(z))^{\gamma_i} v'(z+\zeta)$. Hence this function is, in fact, constant. Thus, if there were an i s.th. $f_i(z)$ is bounded from above, then $v'(z)$ would be bounded away from zero. Then, $f_i(z)$ would be bounded from above for every i. This would contradict the assumption that $\sum_i f_i(z) = z$ for every $z > 0$. Hence, for every i, $f_i(z) \to \infty$ as $z \to \infty$. We can analogously show that for every i, $f_i(z) \to 0$ as $z \to 0$. This also shows that $v'(x) \to \infty$ as $x \to \zeta$ and $v'(x) \to 0$ as $x \to \infty$.

Denote the constant value of $(f_i(z))^{\gamma_i} v'(z+\zeta)$ by κ_i. Then, for every i and j,

$$0 < \theta_i(z) = \frac{f_i(z)}{z} < \frac{f_i(z)}{f_j(z)} = \frac{\left(\dfrac{\kappa_i}{v'(z+\zeta)}\right)^{1/\gamma_i}}{\left(\dfrac{\kappa_j}{v'(z+\zeta)}\right)^{1/\gamma_j}} = \frac{\kappa_i^{1/\gamma_i}}{\kappa_j^{1/\gamma_j}} \left(v'(z+\zeta)\right)^{1/\gamma_j - 1/\gamma_i}.$$

If $\gamma_i < \max_{i=1,\dots,I} \gamma_i = \gamma_j$, then $1/\gamma_j - 1/\gamma_i < 0$. Since $v'(z+\zeta) \to \infty$ as $z \to 0$, the far right-hand side of the above equality converges to 0 as $z \to 0$. Hence $\theta_i(z) \to 0$ as $z \to 0$. We can analogously show that for every i, if $\gamma_i > \min_{i=1,\dots,I} \gamma_i$, then $\theta_i(z) \to 0$ as $z \to \infty$. The limiting behavior of RAA_ϕ follows. □

We now explain the qualitative features of the graph of the shares θ_i as a function of the aggregate certainty equivalent.

Part 1(b) of Proposition 6 implies that, as we move from worse to better models, a consumer whose relative ambiguity aversion is greater (smaller) than that of the representative consumer around $c^{\mathbf{P}}$ will see their share decrease (resp. increase) for models with certainty equivalents marginally greater than $c^{\mathbf{P}}$ as shown in Figure 4.

Consider consumer I with the largest relative ambiguity aversion in the economy. By part 2 of Proposition 6, their relative ambiguity aversion is greater than that of the representative consumer (at all $c^{\mathbf{P}}$). By part 1(b)

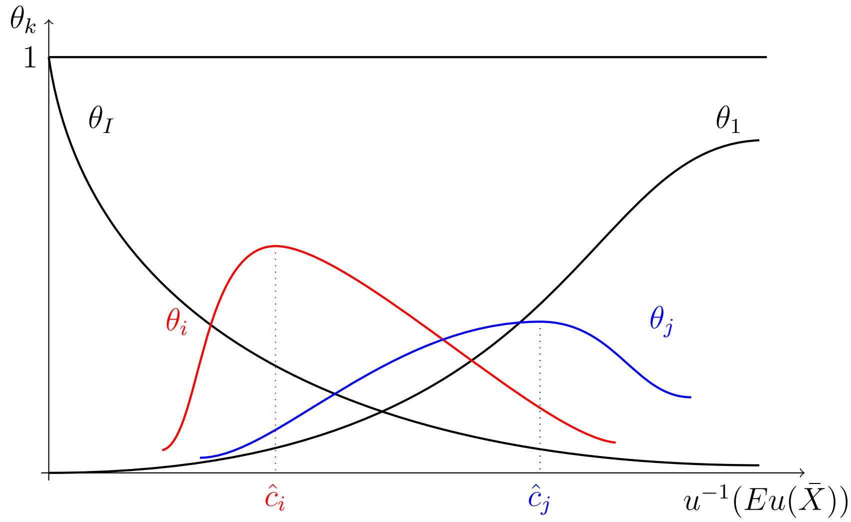

Figure 4: Comparing consumption shares θ_k under Condition 2. Consumer I (resp. 1) is the most (resp. the least) relatively ambiguity averse. i is more relatively ambiguity averse than consumer j.

of Proposition 6, θ_I will be negatively sloped everywhere. Analogously, consumer 1, with the lowest relative ambiguity aversion in the economy, will have a θ_1 that is positively sloped everywhere. From 1 of Proposition 12, the most relatively ambiguity averse consumers get all of $\bar{X} - \zeta$ at the worst models. Therefore, at these models the representative consumer's relative ambiguity aversion is $\max_{i=1,\ldots,I} \gamma_i - \alpha$. Hence, by part 1(b) of Proposition 6, any consumer i with relative ambiguity aversion less than $\max_{i=1,\ldots,I} \gamma_i - \alpha$ will have their share increasing at least initially. Since the representative consumer has decreasing relative ambiguity aversion, we will reach a model, identified by $\hat{c}_i$ in Figure 4, where the representative consumer's relative ambiguity aversion falls below i's; hence, i's share is decreasing to the right of $\hat{c}_i$. For a consumer j relatively less ambiguity averse than i, the representative consumer's ambiguity aversion has to decrease further before j's share peaks. Hence, $\hat{c}_j$ is to the right of $\hat{c}_i$. Taken together, the most relatively ambiguity averse consumers get protected with extra shares at the worst models, the "middling" relative ambiguity averse consumers get extra shares at the "middling" models and the least relatively ambiguity averse ones get compensated by extra shares at the best models.

Finally, note that if $\gamma_i - \alpha = \gamma_j - \alpha$ for all $i, j \in I$, then the efficient allocation would be the same as if all consumers were expected utility consumers: for all i, θ_i is a constant function.

F Strict log-supermodularity

In this Appendix, we give a general result on strict log-supermodularity (SLSPM for short) from which part 2 of Proposition 7 and part 2 of Proposition 8 can be derived.

Let N be a positive integer. For each $x = (x_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$ and each $y = (y_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$, we write $x \geq y$ when $x_n \geq y_n$ for every n. We also write $x \vee y = (\max\{x_n, y_n\})_{n=1,2,\ldots,N}$ and $x \wedge y = (\min\{x_n, y_n\})_{n=1,2,\ldots,N}$. For each $x = (x_n)_{n=1,2,\ldots,N} \in \mathbb{R}^N$, we write $x_{-N} = (x_n)_{n=1,2,\ldots,N-1} \in \mathbb{R}^{N-1}$. By a slight abuse of notation, we use $\geq$, $\leq$, $\vee$, and $\wedge$ for vectors in $\mathbb{R}^{N-1}$ as well.

Let $f : \mathbb{R}^N \to \mathbb{R}_+$. We say that f is *strictly log-supermodular* (SLSPM for short) if

$$f(x)f(y) < f(x \vee y)f(x \wedge y)$$

for every $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$ unless $x \leq y$ or $x \geq y$. That is, the strict log-supermodularity is a stronger property than the log-supermodularity (LSPM) in that the left-hand side is strictly smaller than the right-hand side. If $x \leq y$ or $x \geq y$, then $\{x, y\} = \{x \vee y, x \wedge y\}$ and the left- and right-hand sides would necessarily be equal. The constraint that neither should hold is needed to exclude this case. If $f(x) > 0$ for every $x \in \mathbb{R}^N$, then f is SLSPM if and only if $\ln f$ is strictly supermodular in the sense of Topkis (1998, Section 2.6.1).

Throughout this Appendix, we assume, for every $f : \mathbb{R}^N \to \mathbb{R}_+$ under consideration, that f is differentiable and $f(x) > 0$ for every $x \in \mathbb{R}^N$.

The first part of the following result is stated in Topkis (1998, Section 2.6.1). The second part can be proved in an analogues manner. The proof is omitted.

Lemma 8. *1. f is LSPM if and only if, for all n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a nondecreasing function of x_m.*

2. f is SLSPM if, for every n and m with $n \neq m$, $\partial \ln f(x)/\partial x_n$ is a strictly increasing function of x_m.

Proposition 13. *Suppose that for all $m < N$ and n, $\partial \ln f(x)/\partial x_m$ is nondecreasing in x_n, and strictly increasing in x_n if $n = N$. Define $g : \mathbb{R}^{N-1} \to$*

$\mathbb{R}_{++}$ *by* $g(x_{-N}) = \int_{\mathbb{R}} f(x_{-N}, x_N)\,\mathrm{d}x_N$ *for every* $x_{-N} \in \mathbb{R}^{N-1}$. *Then* g *is SLSPM.*

The assumptions of this proposition imply that f is LSPM but not that f is SLSPM. In fact, they can be met even when f is not SLSPM. The proposition, thus, implies that g can be SLSPM even when f is not. For a twice continuously differentiable f, they are satisfied if, for every $x \in \mathbb{R}^N$, $\frac{\partial^2}{\partial x_m \partial x_N} \ln f(x) > 0$ for every $m < N$, and $\frac{\partial^2}{\partial x_m \partial x_n} \ln f(x) \geq 0$for all $m < N$ and $n \neq m$.

The following proof method is essentially due to Karlin and Rinott (1980, Theorem 2.1). We only need to take special care of preserving strict inequalities under integration.

Proof of Proposition 13 By Fubini's theorem,

$$\begin{aligned}
&g(x_{-N})g(y_{-N}) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_{-N}, z) f(y_{-N}, w)\,\mathrm{d}w\mathrm{d}z = \int_{\mathbb{R}\times\mathbb{R}} f(x_{-N}, z) f(y_{-N}, w)\,\mathrm{d}(z, w) \\
&= \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z=w\}} f(x_{-N}, z) f(y_{-N}, w)\,\mathrm{d}(z, w) \\
&\quad + \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z<w\}} (f(x_{-N}, z) f(y_{-N}, w) + f(y_{-N}, w) f(x_{-N}, z))\,\mathrm{d}(z, w).
\end{aligned} \tag{43}$$

We can similarly show that

$$\begin{aligned}
&g(x_{-N} \vee y_{-N}) g(x_{-N} \wedge y_{-N}) \\
&= \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z=w\}} f(x_{-N} \vee y_{-N}, z) f(x_{-N} \wedge y_{-N}, w)\,\mathrm{d}(z, w) \\
&\quad + \int_{\{(z,w)\in\mathbb{R}\times\mathbb{R} | z<w\}} (f(x_{-N} \vee y_{-N}, z) f(y_{-N} \wedge y_{-N}, w) \\
&\qquad\qquad + f(x_{-N} \vee y_{-N}, w) f(x_{-N} \wedge y_{-N}, z))\,\mathrm{d}(z, w).
\end{aligned} \tag{44}$$

When $z = w$, $(x_{-N}, z) \vee (y_{-N}, w) = (x_{-N} \vee y_{-N}, z)$ and $(x_{-N}, z) \wedge (y_{-N}, w) = (x_{-N} \wedge y_{-N}, w)$. Since f is LSPM,

$$f(x_{-N}, z) f(y_{-N}, w) \leq f(x_{-N} \vee y_{-N}, z) f(x_{-N} \wedge y_{-N}, w).$$

Thus, the first term of the right-hand side of (43) is less than or equal to that of (44). To compare the second terms, assume that $z < w$ and that it is false that $x_{-N} \leq y_{-N}$. Write

$$A(z,w) = f(x_{-N}, z)f(y_{-N}, w), \qquad C(z,w) = f(x_{-N} \vee y_{-N}, z)f(y_{-N} \wedge y_{-N}, w),$$
$$B(z,w) = f(x_{-N}, w)f(y_{-N}, z), \qquad D(z,w) = f(x_{-N} \vee y_{-N}, w)f(x_{-N} \wedge y_{-N}, z).$$

Note first that

$$\begin{aligned}A(z,w)B(z,w) &= (f(x_{-N}, z)f(y_{-N}, z))\,(f(x_{-N}, w)f(y_{-N}, w))\\ &\leq (f(x_{-N} \vee y_{-N}, z)f(x_{-N} \wedge y_{-N}, z))\,(f(x_{-N} \vee y_{-N}, w)f(y_{-N} \wedge y_{-N}, w))\\ &= C(z,w)D(z,w).\end{aligned}$$

Next, without loss of generality, we can assume that there is an M with $1 \leq M < N$ s.th. $x_n > y_n$ if and only if $n \leq M$. Then,

$$\begin{aligned}x_{-N} \vee y_{-N} &= (x_1, \dots, x_M, y_{M+1}, \dots, y_{N-1}),\\ x_{-N} \wedge y_{-N} &= (y_1, \dots, y_M, x_{M+1}, \dots, x_{N-1}).\end{aligned}$$

Moreover,

$$x_{-N} - x_{-N} \wedge y_{-N} = x_{-N} \vee y_{-N} - y_{-N} = (x_1 - y_1, \dots, x_M - y_M, 0, \dots, 0).$$

Denote this by v. For each $m \leq M$, write $v^m = (x_1 - y_1, \dots, x_m - y_m, 0, \dots, 0)$. Then $v^M = v$, $v^0 = 0$, and $v^m - v^{m-1} = (0, \dots, 0, x_m - y_m, 0, \dots, 0)$. Write $h = \ln f$. Then, for every $m \leq M$

$$\begin{aligned}&h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z)\\ =&\int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, x_{M+1} \dots, x_{N-1}, z)\,\mathrm{d}r,\\ &h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w)\\ =&\int_{y_m}^{x_m} \frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, y_{M+1} \dots, y_{N-1}, w)\,\mathrm{d}r.\end{aligned}$$

Since $\partial h/\partial x_m$ is nondecreasing in x_n with $n = M+1, \dots, N-1$ and strictly increasing in x_N,

$$\begin{aligned}&\frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, x_{M+1} \dots, x_{N-1}, z)\\ <&\frac{\partial h}{\partial x_m}(x_1, \dots, x_{m-1}, r, y_{m+1}, \dots, y_M, y_{M+1} \dots, y_{N-1}, w)\end{aligned}$$

for every r. Thus,

$$h(x_{-N} \wedge y_{-N} + v^m, z) - h(x_{-N} \wedge y_{-N} + v^{m-1}, z) < h(y_{-N} + v^m, w) - h(y_{-N} + v^{m-1}, w).$$

Since $x_{-N} \wedge y_{-N} + v^M = x_{-N}$ and $y_{-N} + v^M = x_{-N} \vee y_{-N}$, by taking the summation of each side over $m \leq M$, we obtain

$$h(x_{-N}, z) - h(x_{-N} \wedge y_{-N}, z) < h(x_{-N} \vee y_{-N}, w) - h(y_{-N}, w).$$

That is, $A(z, w) < D(z, w)$. By swapping the roles of x_{-N} and y_{-N} (while maintaining that $z < w$), we can show that $B(z, w) < D(z, w)$.

Since $A(z, w)B(z, w) \leq C(z, w)D(z, w)$, $A(z, w) < D(z, w)$, $B(z, w) < D(z, w)$, and

$$\begin{aligned} &(C(z,w) + D(z,w)) - (A(z,w) + B(z,w)) \\ =& \frac{1}{D(z,w)} \left((C(z,w)D(z,w) - A(z,w)B(z,w)) + (D(z,w) - A(z,w))(D(z,w) - B(z,w)) \right), \end{aligned}$$

we have $A(z, w) + B(z, w) < C(z, w) + D(z, w)$. Since the second term of the right-hand side of (43) is nothing but the integral of $A(z, w) + B(z, w)$ on $\{(z, w) \in \mathbb{R} \times \mathbb{R} \mid z < w\}$ and that of (44) is nothing but the integral of $C(z, w) + D(z, w)$ on the same domain, this completes the proof. □

This proposition can be extended to the case in which the domain of the function is $X_1 \times X_2 \times \cdots \times X_N$, where X_n is an interval in $\mathbb{R}$ for every n.

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