Alpha-maxmin as an aggregation of two selves*

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Abstract

This paper offers a novel perspective on the α-maxmin model, taking its components as originating from distinct selves within the decision maker. Drawing from the notion of multiple selves prevalent in inter-temporal decision-making contexts, we present an aggregation approach where each self possesses its own preference relation. Contrary to existing interpretations, these selves are not merely a means to interpret the decision maker's overall utility function but are considered as primitives. Through consistency requirements, we derive an α-maxmin representation as an outcome of a convex combination of the preferences of two distinct selves. We first explore a setting involving objective information and then move on to a fully subjective derivation.
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1 Introduction

Schmeidler's breakthrough (Schmeidler (1989)) opened the door to a sound, axiomatic foundation of behavior under uncertainty that does not reduce to subjective expected utility and accounts for a non-neutral attitude towards ambiguity. As a pioneering model, the Choquet Expected Utility model received a great deal of attention both with regard to axiomatic characterizations (e.g., Wakker (1990), Chateauneuf (1991), Sarin and Wakker (1992), Chateauneuf (1994), Chew and Karni (1994), Chateauneuf and Tallon (2002), Zhang (2002) Bastianello and Faro (2023)) as well as to its consequences to fundamental economic models (e.g., Dow and Werlang (1992), Epstein and Wang (1994), Epstein and Wang (1995), Dow and Werlang (1994), Marinacci (2000), Chateauneuf et al. (2000), Billot et al. (2000)). Almost simultaneously, the publication of the multiple prior model by

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Gilboa and Schmeidler (1989) had a great impact by elaborating an axiomatic foundation to the related and somewhat more "intuitive" multiple prior model. This approach gave rise to a substantial amount of literature with a wide range of axiomatic developments and applications,[1] while significant criticisms also emerged. Among those was the fact that the multiple prior model, a.k.a the maxmin expected utility model, is widely classified as a strongly paranoiac decision rule because of the embedded min operator – even though the set of priors over which the minimum is taken is part of the representation and thus also reflects the decision maker's attitude towards uncertainty (e.g., Siniscalchi (2009)).

With the goal to consider less extreme attitudes towards uncertainty (but still presenting the min operator in their representations), the models known as variational preferences of Maccheroni et al. (2006) and confidence preferences of Chateauneuf and Faro (2009) emerged as generalizations of the maxmin EU model where each prior representing the set of beliefs is weighted by a kind of degree of plausibility for the decision-maker. It is interesting to note that the intersection of these models is exactly the maxmin EU model. These models are special cases of uncertainty-averse preferences characterized by Cerreia-Vioglio et al. (2011), a very general class of preferences with a representation that makes use of the min operator over priors and also includes the popular smooth model of Klibanoff et al. (2005).

From a different perspective, the task of characterizing less extreme attitudes towards uncertainty was also tackled by Gajdos et al. (2008). It was done by assuming that the decision-maker has some (partial but) objective information about the problem at hand. Gilboa et al. (2010) developed the objective/subjective rationality framework that makes explicit how "cautious" a maxmin decision maker is relative to a set of priors capturing "objective rationality" –see also Echenique et al. (2022), Ceron and Vergopoulos (2022), Faro and Lefort (2019), Bastianello et al. (2022), Frick et al. (2022).

Another strand of literature, to which the current paper is contributing, studies a natural generalization of the maxmin expected utility model –the so-called α-maxmin expected utility model– in which both the minimum and the maximum expected utility over some set of priors are taken into account. The α-maxmin criterion was introduced initially by Hurwicz in an unpublished paper and appeared later as a special case in Arrow and Hurwicz (1972). This criterion is widely used in models formulated in different frameworks. For instance, if the decision-maker faces a possibly coarsely specified decision problem, including unforeseen contingencies, Ghirardato (2001) characterizes a model *à la* Arrow-Hurwicz with non-additive beliefs, echoing previous work by Jaffray (1989). In the context of decision-making under uncertainty with no state space, Olszewski (2007) provides an axiomatic foundation for the counterpart of the α-maxmin EU model for preferences over sets of lotteries, offering a natural concept of objective ambiguity dispensing with state space. Ghirardato et al. (2004) provides an axiomatic in a standard Anscombe-Aumann setting. This seminal paper also generated a sizable literature pointing to issues concerning the foundations of the model (Eichberger et al. (2011)) or the identification of the pessimism index together with the set of prior (Siniscalchi (2009), Klibanoff et al. (2014), Klibanoff

[1]To cite a few from the axiomatic perspective for static decisions: Chateauneuf (1991), Casadesus-Masanell et al. (2000), Ghirardato et al. (2003), Alon and Schmeidler (2014).

et al. (2022), Chateauneuf et al. (2023), Hartmann (2023)), as well as extensions such as the neo-additive capacity approach of Chateauneuf et al. (2007)), or the dual-self representation of Chandrasekher et al. (2022) and Mononen (2024). Another representation combining the max and min operators that generalizes the maxmin expected utility was proposed by Casaca et al. (2014).

While the α-maxmin EU model has received much attention in the decision theory literature, its use in economic applications has been more scarce. Beissner et al. (2020) provide an axiomatization of a dynamically consistent version of the model and apply it to the CAPM. Beissner and Werner (2023) study risk-sharing under various (possibly non-convex) preferences.

In this paper we contribute to the existing literature by examining the α-maxmin model from a slightly different perspective. We interpret the two components of the criterion, the max and the min, as originating from two distinct selves, with the α-maxmin criterion serving as a means to aggregate these two selves. The notion that an individual comprises multiple selves is a common modeling approach, particularly in inter-temporal decision-making contexts, where different selves make decisions at different points in time. While the concept of multiple selves co-existing at a single point in time is also present in static decision-making under uncertainty (as seen in Chandrasekher et al. (2022)), it is often more of an interpretation of the functional form rather than a foundational element of the model. In contrast, we posit the existence of these selves, each with its own preference relation, and explore methods of combining them to form a final preference. The outcome is an aggregation rule that produces an α-maxmin representation, which can be conceptualized as the result of an internal (though unmodeled) deliberation process between the two selves.

More precisely, we provide a simple axiomatization of a decision-maker who has to cope with her two selves, one optimistic (adventurous) and one pessimistic (cautious). Each self has thus its own preference relation that has to be aggregated through some consistency requirements to "yield" the final preference. The selves' preferences are not sub-relations of the final preferences except in the limit cases. We derive an α-maxmin representation (and generalization thereof) as the result of a convex combination between the preferences of the two selves, where α is, in some sense, the "bargaining" weight of the pessimistic self and $(1-\alpha)$ the bargaining weight of the optimistic self.

The preference relation of each self cannot be directly observed through choices; only choices reflecting the "final" preference are observable, aligning with the objective-subjective approach.[2] Two interpretations can be suggested for these selves' preferences and their observability or lack thereof. The first interpretation posits that these preferences could represent those the decision maker uses when providing advice, such as giving financial guidance to a cautious or an adventurous investor. She herself has her own attitude, but when advising others, she tries to separate the "hard" information she has from her tastes.

[2]It is worth noting that we have two such relations here, while Gilboa et al. (2010) require only one. While it would be possible to formally conduct a similar analysis with only one underlying (non-observable through choices) preference relation, a maybe more significant distinction from Gilboa et al. (2010) is that the cautious and adventurous selves are not subsidiary relations of the final preference within our framework. We elaborate on the relationship of our construction with Gilboa et al. (2010) and Frick et al. (2022) in Section 3.

So, the two "selves" relations are observable from her advice to others, while her own relation is observable from her own choices. The second interpretation regards these selves as deliberation tools used by the decision maker who seeks to consider two "extreme" points of view before making a decision. However, these viewpoints must adhere to rational axioms. Thus, the axiomatic construction proposed in this paper can be understood as a normative way of building a moderate preference relation that aggregates extreme views.

We begin by examining scenarios where some objective, albeit partial, information about relative likelihood is accessible. This typically manifests as a set of probability distributions on the state space, which both selves accept at face value. Examples illustrate that this situation naturally arises when the core of a convex capacity can represent probabilistic information. Additionally, the Choquet case facilitates the derivation of an α-maxmin rule with conditional preferences. Next, we move to a fully subjective setting and assume each self is either of the maxmin or of the maxmax type. Within this setting, consistency necessitates that both selves share an identical collection of priors. Lastly, we propose a generalization that does not rely on a specific functional form for each self, whose preferences need not be dual to one another and show how it can be used to derive a no-trade interval à la Dow and Werlang (1992). Interestingly, the case $\alpha = 1/2$ coincides, in this construction, with ambiguity neutrality.

The paper is constructed as follows. Section 2 introduces the framework and the necessary background and definitions. Section 3 derives the $\alpha-$Choquet and $\alpha-$maxmin rules when the selves have access to a common capacity or a common set of probabilities representing the information available. This section also contains a derivation of the conditional maxmin model and two examples. Section 4 retains the Choquet and Maxmin case but provides the analysis in a fully subjective setting. Section 5 goes beyond this and assumes general preferences. Proofs are gathered in the Appendix.

2 Framework

We consider a Savage-type model with monetary payoffs similar to Chateauneuf (1991), under the assumption of a finite state space S. In our framework, an act is a real-valued function defined on S. Let F denote the set of all acts. The set F of acts is equipped with the natural (euclidean) topology. For $f, g \in F$, we write $f \geq g$ if $f(s) \geq g(s)$ for all $s \in S$ and $f > g$ if $f(s) > g(s)$ for all $s \in S$. For each $E \subseteq S$ and $f, g \in F$, $f_E g \in F$ denote the element of F equal to f over E and to g outside E. The constant act whose image is the singleton $\{x\}$ is denoted by x.

Two acts $f, g \in F$ are said to be

- *comonotonic* if $(f(s) - f(s'))\,(g(s) - g(s')) \geq 0$ for all $s, s' \in S$,
- *complementary* if $f(s) + g(s) = f(s') + g(s')$ for all $s, s' \in S$.

Consider a functional I from F to $\mathbb{R}$. We say that I is *monotonic* if $I(f) \geq I(g)$ for all $f, g \in F$ such that $f(s) \geq g(s)$ for all $s \in S$. We say that it is *constant additive* if $I(f + x) = I(f) + x$ for all $f \in F$ and $x \in \mathbb{R}$. We say that it is *positively homogeneous* if

$I(\gamma f) = \gamma I(f)$ for all $\gamma \geq 0$ and $f \in F$. It is *constant linear* if it is constant additive and positively homogeneous.

A capacity v on S is a function from the power set of S to $[0,1]$ satisfying $v(\emptyset) = 0$, $v(S) = 1$ and $v(E) \geq v(F)$ for all $E, F \subseteq S$ such that $F \subseteq E$. A capacity v on S is said to be a probability if it is additive, that is, if $v(E \cup F) = v(E) + v(F)$ for all $E, F \subseteq S$ such that $E \cap F = \emptyset$. It is said to be *convex* if $v(E \cup F) + v(E \cap F) \geq v(E) + v(F)$ for all $E, F \subseteq S$ and concave if $v(E \cup F) + v(E \cap F) \leq v(E) + v(F)$ for all $E, F \subseteq S$. For every capacity v on S, we define the dual capacity $\overline{v}$ on S by setting $\overline{v}(E) = 1 - v(E^c)$ for all $E \subseteq S$. Consider a capacity v on S. The Choquet integral of $f \in F$ with respect to v is defined by

$$\int_S f(s)dv(s) \;=\; \int_{-\infty}^{0} \left[v[\{s \in S,\; f(s) \geq x\}] - 1\right] dx \;+\; \int_0^{+\infty} v[\{s \in S,\; f(s) \geq x\}]dx.$$

Suppose v is a capacity on S. The core $C(v)$ of v is defined as the collection of all probabilities μ on S such that $\mu(E) \geq v(E)$ for all $E \subseteq S$. If v is convex, then $C(v)$ is nonempty, and we have for all $f \in F$:

$$\int_S f(s)dv(s) \;=\; \min_{\mu \in C(v)} \int_S f(s)d\mu(s).$$

If v is concave, then $C(\overline{v})$ is nonempty, and we have for all $f \in F$:

$$\int_S f(s)dv(s) \;=\; \max_{\mu \in C(\overline{v})} \int_S f(s)d\mu(s).$$

Consider $\alpha \in [0,1]$, a closed and convex set C of probabilities on S and a capacity v on S. We define $I_{\alpha,C}$ as the real-valued functional on F such that, for all $f, g \in F$,

$$I_{\alpha,C}(f) \;=\; \alpha \min_{\mu \in C} \int_S f(s)d\mu(s) + (1-\alpha) \max_{\mu \in C} \int_S f(s)d\mu(s).$$

We define $I_{\alpha,v}$ as the real-valued functional on F such that, for all $f, g \in F$,

$$I_{\alpha,v}(f) \;=\; \alpha \min \int_S f(s)dv(s) + (1-\alpha) \max \int_S f(s)d\overline{v}(s).$$

Consider a binary relation $\succsim'$ on F and a real-valued functional I defined on F. We say that I is a representation of $\succsim'$ if, for all $f, g \in F$, $f \succsim' g \iff I(f) \geq I(g)$. We say that a pair (α, C) provides an α-maxmin representation of $\succsim'$ if $I_{\alpha,C}$ is a representation of $\succsim'$. When $\alpha = 1$, we speak of a maxmin representation while, when $\alpha = 0$, we speak of a maxmax representation. Finally, we say that a pair (α, v) provides an α-Choquet representation of $\succsim'$ if $I_{\alpha,v}$ is a representation of $\succsim'$.

Say that a convex capacity v on S is *regular* (see Chateauneuf et al. (2011)) if,

$$\forall A, B \subset S, A, B \neq \emptyset, 0 < v(A \cap B), v(A \cup B) < 1 \Rightarrow v(A \cap B) + v(A \cup B) = v(A) + v(B) \quad (1)$$

Finally, conditionally on $E \subseteq S$ being realized, one can define, for any $A \subseteq E$ the conditional capacity $v_E(A) = \frac{v(A)}{v(A)+1-v(A \cup \bar{E})}$. When v is convex and regular, v_E is convex and $C(v_E) = \{P_E | P \in C(v)\}$, where P_E denotes the conditional probability measure P given E.

3 Objective information

In this section, we assume a closed and convex set C of probabilities on S representing the objective but partial information available to the DM (and her two selves). This approach is related to Gajdos et al. (2008), although we consider preferences defined over acts f in F. The exogenously given set of priors C offers a natural concept of objective ambiguity in a Savage-type model. Objective ambiguity was also previously modeled under the assumption of intractable states of nature by Olszewski (2007) and Ahn (2008) with preferences defined over sets of lotteries.

A decision-maker is characterized by three preference relations

- $\succsim_1$ and $\succsim_2$ on F representing her pessimistic and optimistic selves respectively,
- $\succsim$ on F representing her observable behavior.

We assume that both selves are faced with the objective information described by the set of priors C, thereby imposing a consistency requirement among the two selves. We begin with axioms on the selves' preferences. **A1** states that the two selves have transitive preferences; completeness is not assumed, being a property that will be satisfied as a consequence of the combination of transitivity with the others introduced below.

A1 $\succsim_1$ and $\succsim_2$ are transitive.

The next axiom, **A2**, imposes that the two selves evaluate constant acts in the same way, according to the natural order on $\mathbb{R}$.

A2 For all $x, y \in \mathbb{R}$, $x \succsim_1 y$ iff $x \succsim_2 y$ iff $x \geq y$. Axioms **A3** and **A4** compare expected utility of random variables with respect to priors in the set C to (the utility of) constant acts. They deliver the fact that $\succsim_1$ is of the pessimistic (min) type while $\succsim_2$ is of the optimistic (max) type, both with respect to the given set C.

A3 For all $f \in F$ and $x \in \mathbb{R}$, (i) if $\int_S f(s)d\mu(s) \geq x$ for all $\mu \in C$, then $f \succsim_1 x$ and (ii) if $x \geq \int_S f(s)d\mu(s)$ for all $\mu \in C$, then $x \succsim_2 f$.

A4 For all $f \in F$ and $x \in \mathbb{R}$, (i) if $x \geq \int_S f(s)d\mu(s)$ for some $\mu \in C$, then $x \succsim_1 f$ and (ii) if $\int_S f(s)d\mu(s) \geq x$ for some $\mu \in C$, then $f \succsim_2 x$. Next, we impose axioms on "final" preferences, which are assumed complete, transitive and continuous.

B1 $\succsim$ is complete and transitive.

B2 For all $f \in F$, $\{g \in F,\ g \succsim f\}$ and $\{g \in F,\ f \succsim g\}$ are closed in F.

The next axiom, **B3**, is a form of constant-additivity. Scaling indifferent acts and adding constants do not change the preference.

B3 For all $f, g \in F$, $x \in \mathbb{R}$ and $\gamma \geq 0$, if $f \sim g$, then $\gamma f + x \sim \gamma g + x$.

Finally, **B4** encapsulates the fact that $\succsim$ follows unanimity of the two selves.

B4 For all $f, g \in F$, if $f \succsim_1 g$ and $f \succsim_2 g$, then $f \succsim g$. It is similar to Axiom 7 in

Ghirardato et al. (2004). It generalizes the axiom of Caution in Gilboa et al. (2010) and corresponds to the Security-Potential Dominance axiom in Frick et al. (2022).

The preceding axioms characterize an $\alpha-$maxmin decision-maker. As the following proposition demonstrates, the aggregation between the two selves is simply to take a convex combination of the maximal and the minimal expectation (with respect to the set C).

Proposition 1 *$(\succsim_1, \succsim_2)$ satisfies* **A1**–**A4** *and $\succsim$ satisfies* **B1**–**B4** *if and only if there exists $\alpha \in [0,1]$ such that (α, C) provides an α-maxmin representation of $\succsim$. Moreover, α is unique if C is non-singleton.*

Proposition 1 is comparable in spirit to the results of Ghirardato et al. (2004), Gilboa et al. (2010) and Frick et al. (2022). The maxmin and maxmax representations are obtained as a direct consequence of the combination of **A3** and **A4**, and the α-maxmin representation is obtained through arguments similar to those of Ghirardato et al. (2004) and Frick et al. (2022). Ghirardato et al. (2004) assume a single preference relation representing the agent's behavior. From it, they derive an auxiliary preference relation, the so-called unambiguous preference. This latter preference admits a unanimity representation à la Bewley (1986, 2002), and hence gives rise to pessimistic (maxmin) and optimistic (maxmax) dual evaluations in a natural way. An axiom of consistency with respect to these dual evaluations delivers the α-maxmin representation of the initial preference.

Gilboa et al. (2010) start with two preference relations representing subjective and objective rationality from the outset. The axioms impose a unanimity representation of the objective rationality preference. Consistency requirements between the two forms of rationality, including a form of caution, deliver the maxmin representation of subjective rationality. In our terms, the pessimistic self is thus, through caution, assigned all the bargaining power.

Frick et al. (2022) extend the analysis of Gilboa et al. (2010) by allowing each self to receive some bargaining power. Objective rationality still has a unanimity representation leading, as in Ghirardato et al. (2004), to pessimistic (maxmin) and optimistic (maxmax) dual evaluations. Consistency with respect to the latter delivers the α-maxmin representation of subjective rationality. Our analysis departs from that of Frick et al. (2022) in the assumption of exogenously given pessimistic and optimistic preferences. It is further simplified by the assumption of an exogenously given set of probabilities representing the objective information, an assumption that we dispense with in the next section.

Next, we extend the analysis to conditional preferences, in which the relevant set of distributions is the set of all updated distributions (i.e., the so-called "full-Bayes update" of C). In the next result, we fix $E \subseteq S$ such that $\mu(E) > 0$ for all $\mu \in C$ and consider binary relations $\succsim_1^E$, $\succsim_2^E$ and $\succsim^E$ on F. Here, $\succsim_1^E$ and $\succsim_2^E$ represent the pessimistic and optimistic preferences conditional on E, respectively, while $\succsim^E$ represents the final preferences conditional on E.

The next axiom collects different standard requirements that we need to impose on the selves' conditional preferences to derive our next result:

A5 (i) $\succsim_1^E$ and $\succsim_2^E$ are transitive.

(ii) For all $x, y \in \mathbb{R}$, $x \succsim_1^E y$ iff $x \succsim_2^E y$ iff $x \geq y$.
(iii) For all $f \in F$ and $x \in \mathbb{R}$, $f \sim_1^E x$ iff $f_E x \sim_1 x$ and $f \sim_2^E x$ iff $f_E x \sim_2 x$.
We will also use the following conditional versions of Axioms **B1–B4**.

CB1 $\succsim^E$ is complete and transitive.

CB2 For all $f \in F$, $\{g \in F,\, g \succsim^E f\}$ and $\{g \in F,\, f \succsim^E g\}$ are closed in F.

CB3 For all $f, g \in F$, $x \in \mathbb{R}$ and $\gamma \geq 0$, if $f \sim^E g$, then $\gamma f + x \sim^E \gamma g + x$.

CB4 For all $f, g \in F$, if $f \succsim_1^E g$ and $f \succsim_2^E g$, then $f \succsim^E g$.

Proposition 2 *$(\succsim_1, \succsim_1^E, \succsim_2, \succsim_2^E)$ satisfies* **A1–A5** *and $\succsim^E$ satisfies* **CB1–CB4** *if and only if there exists $\alpha_E \in [0,1]$ such that (α_E, C_E) provides an α-maxmin representation of $\succsim^E$. Moreover, α_E is unique if C_E is non-singleton.*

Proposition 2 echoes the result in Faro and Lefort (2019) who provide a dynamic version of the Gilboa et al. (2010) model in which unconditional beliefs are updated prior-by-prior.

Corollary 1 *Suppose* **A1–A4** *and* **B1–B4** *hold. If C is the core of a convex capacity v, there exists $\alpha \in [0,1]$ such that (α, v) provides an α-Choquet representation of $\succsim$. Moreover, α is unique if v is not a probability distribution (i.e., if $C(v)$ is not a singleton).*

Corollary 2 *Suppose* **A1–A5** *and* **CB1–CB4** *hold. If C is the core of a convex, regular capacity v, there exists $\alpha_E \in [0,1]$ such that (α_E, v_E) provides an α-Choquet representation of $\succsim_E$. Moreover, α_E is unique if v_E is not a probability distribution (i.e., if $C(v_E)$ is not a singleton).*

As stated in the Introduction, it is easy to come up with examples where objective information comes in the form of the core of a capacity. An example of this is when information is given in the form of bounds for singleton. Assume that the decision maker is told that the probability $p(s)$ of state s is such that $p(s) \in [a_s, b_s]$ for all $s \in S$, with $b_s \geq a_s$ for all s and $\sum_s b_s \geq 1 \geq \sum_s a_s$. De Campos et al. (1994) show that the set of such distributions is actually the core of the convex capacity v defined by : for each $E \subseteq S$, $v(E) = \max(\sum_{s \in E} a_s, 1 - \sum_{s \notin E} b_s)$. If the DM conforms to **A1–A4** with $C = \text{core}(v)$ and **B1–B4**, then (α, v) is an $\alpha-$Choquet representation of her preferences.

To identify α, one needs to elicit the certainty equivalent $\gamma(E)$ of some event E and compute $\alpha = \frac{\gamma(E) - (1 - v(\bar{E}))}{v(E) - (1 - v(\bar{E}))}$. Obviously, if one reveals through an experiment that α thus defined depends on E, this would reveal that the decision-maker is not of the $\alpha-$Choquet type.

This example can also be used to illustrate Corollary 2. The capacity v defined above satisfies property (1) for instance whenever $b_s = 1$ for all s, or $a_s = 0$ for all s or, more generally, if $\sum_{s \in E} a_s \geq 1 - \sum_{s \notin E} b_s$ for all E or if $\sum_{s \in E} a_s \leq 1 - \sum_{s \notin E} b_s$ for all E. In that case, it is regular and hence, as established by Chateauneuf et al. (2011), the conditional preferences $\succsim_E$ of a decision-maker satisfying **A1–A5** with $C = \text{core}(v)$ and **CB1–CB4**, admit an $\alpha-$Choquet representation (α_E, v_E).

Another (class of) example(s) is the case of "inner probabilistic information". This arises when there is an objective probability on a sub-algebra $\mathcal{A}$ of 2^S. Denote P_0 this probability and let $v(E) = \max\{P(E); P \text{ is a probability on } (S, 2^S) \text{ s.th. } P = P_0 \text{ on } \mathcal{A}\}$, while $\bar{v}(E) = \min\{P(E); P \text{ is a probability on } (S, 2^S) \text{ s.th. } P = P_0 \text{ on } \mathcal{A}\}$. Classical results show that v is the inner probability of P_0 on $\mathcal{A}$, i.e., $v(E) = \inf_{P \in \mathcal{P}}\{P(E)\}$ where $\mathcal{P} = \{P \text{ on } (S, 2^S) \text{ s.th. } P = P_0 \text{ on } \mathcal{A}\}$. Furthermore, v thus defined is convex. If the decision-maker satisfies **A1**–**A4** with $C = \mathrm{core}(v)$ and **B1**–**B4**, her preferences can be represented by $I(f) = \alpha \min_{P \in \mathcal{P}} \int f dP + (1-\alpha) \max_{P \in \mathcal{P}} \int f dP$, according to Corollary 1, that is, $I(f) = \alpha \int_S f dv(s) + (1-\alpha) \int_S f d\bar{v}(s)$.

4 A fully subjective derivation

In this section, we no longer assume an exogenously given set C representing the objective probabilistic information and characterize α-maxmin and α-Choquet representations. Axioms **A3** and **A4** are now void since there is no exogenous set C one can use to express pessimism and optimism. We thus impose C-independence as well as ambiguity aversion (resp. loving) on $\succsim_1$ (resp. $\succsim_2$). Furthermore, there is no longer an exogenous coordination device among the two selves and we need an extra axiom to ensure that the subjective sets of priors of the two selves coincide. The axioms relating the two selves to $\succsim$ remain unchanged.

A binary relation $\succsim'$ on F is standard if it is complete, transitive, continuous and monotonic in the following sense:

(1) For all $f, g \in F$, $f \succsim' g$ or $g \succsim' f$.

(2) For all $f, g, h \in F$, if $f \succsim' g$ and $g \succsim' h$, then $f \succsim' h$.

(3) For all $f \in F$, $\{g \in F,\ g \succsim' f\}$ and $\{g \in F,\ f \succsim' g\}$ are closed in F.

(4) For all $f, g \in F$, if $f \geq g$, then $f \succsim' g$ and if $f > g$, then $f \succ' g$.

We say that the pair $(\succsim_1, \succsim_2)$ is standard if each of $\succsim_1$ and $\succsim_2$ is standard. We say that it is standard* if, in addition, each of $\succsim_1$ and $\succsim_2$ is positively homogeneous and constant additive in the following sense:

(5) For all $f, g \in F$, $x \in \mathbb{R}$ and $\gamma \geq 0$, if $f \sim' g$, then $\gamma f + x \sim' \gamma g + x$.

We continue with more axioms on the selves' preferences. Axiom **A6** is related to Axioms 8 and 9 in Echenique et al. (2022). It is instrumental to obtain, in the representation given by Proposition 3, a single set of priors C that represent both the maxmin self and the maxmax self.

A6 For all $i \in \{1, 2\}$ and complementary $f, g \in \mathcal{F}$, $f + g \sim_i f$ if and only if $g \sim_{-i} \mathbf{0}$.

Axiom **A7** imposes a maxmin form on $\succsim_1$ and a maxmax form on $\succsim_2$.

A7 For all $f, g \in F$ and $\gamma \in [0, 1]$, (i) if $f \sim_1 g$, then $\gamma f + (1-\gamma) g \succsim_1 f$ and (ii) if $f \sim_2 g$, then $f \succsim_2 \gamma f + (1-\gamma) g$.

Proposition 3 $(\succsim_1, \succsim_2)$ *is standard* and satisfies* **A6** *and* **A7**, *and* $\succsim$ *satisfies* **B1**–**B4** *if and only if there exists* $\alpha \in [0, 1]$ *and a closed and convex set* C *of probabilities on* S *such that*

(i) *C provides a maxmin representation of $\succsim_1$ and a maxmax representation of $\succsim_2$,*

(ii) *(α, C) provides an α-maxmin representation of $\succsim$.*

Moreover, C is unique, and α is unique if C is non-singleton.

Replacing **A7** with a form of comonotonic independence, **A8** below, yields an α-Choquet representation with a capacity v for $\succsim_1$ and its conjugate for $\succsim_2$.

A8 For all $f, g, h \in F$ with g and h comonotonic, (i) if $f \sim_1 g$, then $f + h \succsim_1 g + h$ and (ii) if $f \sim_2 g$, then $g + h \succsim_2 f + h$.

Proposition 4 *$(\succsim_1, \succsim_2)$ is standard and satisfies* **A6** *and* **A8**, *and $\succsim$ satisfies* **B1**–**B4** *if and only if there exists $\alpha \in [0,1]$ and a convex capacity v on S such that*

(i) *v and $\overline{v}$ provide Choquet representations of $\succsim_1$ and $\succsim_2$ respectively,*

(ii) *(α, v) provides an α-Choquet representation of $\succsim$.*

Moreover, v is unique, and α is unique if v is non-additive.

In the specific context of Proposition 4, **A6** can be replaced with the following simpler condition: For all $\gamma_1, \gamma_2 \in \mathbb{R}$ and $E \subseteq S$, if $1_E 0 \sim_1 \gamma_1$ and $1_E 0 \sim_2 \gamma_2$, then $\gamma_1 + \gamma_2 = 1$.

5 Generalization

In this section, we no longer commit to assumptions implying maxmin or Choquet representations of the selves' preferences and seek for general representation of final preferences. We no longer require that the two selves have the same set of priors, i.e., in the present more general context, the selves' preferences can be represented by functionals that are not dual to one another.

A9 For all $f \in F$ and $x \in \mathbb{R}$, if $f \succsim_1 x$, then $f \succsim_2 x$.

Proposition 5 *$(\succsim_1, \succsim_2)$ is standard* if and only if there exist (unique) monotonic and constant linear functionals I_1 and I_2 from F to $\mathbb{R}$ such that, for all $f, g \in F$,*

$$f \succsim_1 g \iff I_1(f) \geq I_1(g) \quad \text{and} \quad f \succsim_2 g \iff I_2(f) \geq I_2(g).$$

Moreover, $(\succsim_1, \succsim_2)$ satisfies **A6** *if and only if $I_2(f) = -I_1(-f)$ for all $f \in F$ and satisfies* **A9** *if and only if $I_1(f) \leq I_2(f)$ for all $f \in F$.*

B5 For all $i \in \{1,2\}$, $x \in \mathbb{R}$ and $f \in F$, (i) if $f \succsim_i x$ and $x \succ f$, then $y \succsim_{-i} f$ for all $y \in \mathbb{R}$ such that $y \succsim f$, and (ii) if $x \succsim_i f$ and $f \succ x$, then $f \succsim_{-i} y$ for all $y \in \mathbb{R}$ such that $f \succsim y$.

Proposition 6 $(\succsim_1, \succsim_2)$ *is standard* and* $\succsim$ *satisfies* **B1**–**B5** *if and only if there exist (unique) monotonic and constant linear functionals* I_1 *and* I_2 *from* F *to* $\mathbb{R}$ *representing* $\succsim_1$ *and* $\succsim_2$ *respectively and a closed interval* $A \subseteq [0,1]$ *such that, for all* $f, g \in F$,

$$f \succsim g \iff \min_{\alpha \in A} \{\alpha I_1(f) + (1-\alpha) I_2(f)\} \geq \min_{\alpha \in A} \{\alpha I_1(g) + (1-\alpha) I_2(g)\},$$

or a closed interval $A \subseteq [0,1]$ *such that, for all* $f, g \in F$,

$$f \succsim g \iff \max_{\alpha \in A} \{\alpha I_1(f) + (1-\alpha) I_2(f)\} \geq \max_{\alpha \in A} \{\alpha I_1(g) + (1-\alpha) I_2(g)\}.$$

Moreover, A *is unique in each case if there exist* $f, g \in F$ *and* $x, y \in \mathbb{R}$ *such that* $f \succ_2 x$ *and* $x \succsim_1 f$ *while* $g \succ_1 y$ *and* $y \succsim_2 g$.

The representation obtained in Proposition 6 extends Lemma B.5 in Ghirardato et al. (2004) and Lemma A. 1 of Frick et al. (2022). It generalizes the standard $\alpha-$maxmin representation in various ways. First, it does not assume that the selves' preferences are maxmin and maxmax. Second, it does not assume that the functionals representing the two selves are dual to one another. For instance, they could be maxmin and maxmax prefrences, with respect to different sets C_1 and C_2, much as in the asymmetric representation of Chandrasekher et al. (2022). They could also be Choquet with respect to arbitrary capacities. Finally, it does not assume that the agent's final preferences aggregate linearly her selves' preferences.

B6 For all $x \in \mathbb{R}$ and $f \in F$, if $f \succsim_1 x$, then $f \succsim x$ and, if $f \succsim x$, then $f \succsim_2 x$.

Corollary 3 $(\succsim_1, \succsim_2)$ *is standard* and satisfies* **A9**, *and* $\succsim$ *satisfies* **B1**–**B4** *and* **B6** *if and only if there exist (unique) monotonic and constant linear functionals* I_1 *and* I_2 *from* F *to* $\mathbb{R}$ *representing* $\succsim_1$ *and* $\succsim_2$ *respectively and* $\alpha \in [0,1]$ *such that, for all* $f, g \in F$,

$$f \succsim g \iff \alpha I_1(f) + (1-\alpha) I_2(f) \geq \alpha I_1(g) + (1-\alpha) I_2(g).$$

Moreover, α *is unique if there exists* $f \in F$ *and* $x \in X$ *such that* $f \succ_2 x$ *and* $x \succsim_1 f$.

A version of **B6** holds by construction in Ghirardato et al. (2004) as, in our terminology, the selves have maxmin and maxmax preferences with respect to revealed ambiguity. Likewise, a version of **B6** holds in Gilboa et al. (2010) because the selves have maxmin and maxmax preferences with respect to a set appearing in the unanimity representation of the subrelation capturing "objective rationality". A similar point can be made for Frick et al. (2022). In the two latter papers, the fact that the "objective rationality" preference is a subrelation of the agent's preferences is a consequence of the Consistency axiom.

For a possible illustration of the construction in a financial setting, consider that an act $f \in F$ represents the payoff delivered by an asset at the various states. Suppose, as implied by Proposition 5, that $I_2(f) = -I_1(-f)$. This has the following interpretation: the selling (resp. buying) price of the pessimistic self for f equals the buying (resp. selling) price of the optimistic self for f. Suppose also, consistently with **B6**, that $I_1(f) < I_2(f)$. This means that the evaluation of f made by the pessimistic self is lower than that made by the optimistic self. Then, under the assumptions of Corollary 3, there is a (possibly trivial) no-trade interval of prices à la Dow and Werlang (1992), that is, a range of prices at which the agent neither wants to buy the asset nor to sell it short, if and only if $I(f) \leq -I(-f)$ where I is the representing functional defined by $I(g) = \alpha I_1(g) + (1-\alpha) I_2(g)$ for all $g \in F$. This, in turn, is equivalent to $\alpha \geq 1/2$. Hence, and as long as the bargaining weight of the pessimistic self remains higher than that of the optimistic self, Corollary 3 predicts a no-trade interval. Note also that the case where $\alpha = 1/2$ makes this no-trade interval of prices a trivial one. This is similar to what happens under standard subjective expected utility preferences. Hence, it is tempting to think of the case $\alpha = 1/2$ as one of neutrality towards ambiguity. While this has a bit of truth in this specific application, it is known that the case $\alpha = 1/2$ does not correspond to ambiguity neutrality in general. The role of our final result is precisely to clarify the circumstances under which $\alpha = 1/2$ leads to neutrality towards ambiguity.

The next axiom appears in Siniscalchi (2009) under the name Complementary Independence and is known to characterize, in the context of the maxmin model, the central symmetry of the set of priors.

B7 For all $f, \overline{f}, g, \overline{g} \in F$ such that each of $\{f, \overline{f}\}$ and $\{g, \overline{g}\}$ is made of complementary acts, if $f \sim \overline{f}$ and $g \sim \overline{g}$, then $f + g \sim \overline{f} + \overline{g}$.

Corollary 4 *Suppose $(\succsim_1, \succsim_2)$ is standard* and let I_1 and I_2 be functionals as in Proposition 5. Suppose also $(\succsim_1, \succsim_2)$ satisfies* **A6**. *Suppose finally $\succsim$ satisfies* **B1**–**B5**. *If $\succsim$ additionally satisfies* **B7**, *then there exists a (unique) probability measure μ on S such that, for all $f \in F$,*

$$\frac{1}{2} I_1(f) + \frac{1}{2} I_2(f) \ = \ \int_S f(s) d\mu(s).$$

Corollary 4 extends the result in Siniscalchi (2009) to the case of α-maxmin preference and its generalization as per Proposition 6 with $I_2(f) = -I_1(-f)$. It gives a rationale for interpreting $\alpha = \frac{1}{2}$ as reflecting ambiguity neutrality since for that value, the functional form is actually an expected utility.

Appendix

Proof of Proposition 1. Suppose first **A1**–**A4** hold. Fix $f \in F$ and define $x = \min_{\mu \in C} \int_S f(s) d\mu(s)$. We have $\int_S f(s) d\mu(s) \geq x$ for all $\mu \in C$ and, by obtain **A3**, obtain $f \succsim_1 x$. Meanwhile, we have $x \geq \int_S f(s) d\mu(s)$ for some $\mu \in C$ (take a $\mu \in C$ achieving the minimum) and, by obtain **A3**, obtain $x \succsim_1 f$. Overall, we have $f \sim_1 x$. Fix also $g \in F$ and define $y = \min_{\mu \in C} \int_S g(s) d\mu(s)$. By the same argument, we obtain $g \sim_1 y$. Thanks to **A1**, it must be that $f \succsim_1 g$ is equivalent to $x \succsim_1 y$ and, by **A2**, further equivalent to $x \geq y$. This shows that, for all $f, g \in F$, we have

$$f \ \succsim_1 \ g \iff \min_{\mu \in C} \int_S f(s) d\mu(s) \ \geq \ \min_{\mu \in C} \int_S g(s) d\mu(s).$$

A symmetric argument shows that, for all $f, g \in F$,

$$f \ \succsim_2 \ g \iff \max_{\mu \in C} \int_S f(s) d\mu(s) \ \geq \ \max_{\mu \in C} \int_S g(s) d\mu(s).$$

Suppose now that **B1**–**B4** hold. Proceed as in Lemma 1 of Chateauneuf (1994) to obtain a real-valued functional I defined on F such that, for all $f, g \in F$, $f \succsim g$ if and only if $I(f) \geq I(g)$ and such that $I(x) = x$ for all $x \in \mathbb{R}$. (Note, however, that, in the present paper, we use the classical continuity axiom **B2** which allows to construct certainty equivalents through the connexity of F.)

Note that I is monotonic in the following sense: for all $f, g \in F$ such that $f(s) \geq g(s)$ for all $s \in S$, we have $I(f) \geq I(g)$. Indeed, consider such $f, g \in F$. By the representations of $\succsim_1$ and $\succsim_2$ obtained above, we have $f \succsim_1 g$ and $f \succsim_2 g$. Then, **B4** yields $f \succsim g$ and $I(f) \geq I(g)$.

Note also that I is constant linear in the following sense: for all $f \in F$, $x \in \mathbb{R}$ and $\gamma \geq 0$, we have $I(\gamma f + x) = \gamma I(f) + x$. Indeed, let $y = I(f)$ so that $I(f) = I(y)$ and $f \sim y$. Then, **B3** yields $\gamma f + x \sim \gamma y + x$ and $I(\gamma f + x) = I(\gamma y + x) = \gamma y + x = \gamma I(f) + x$.

Thanks to **B4**, we can apply Lemma A.3 from Frick et al. (2022) and obtain the existence of $\alpha \in [0, 1]$ such that, for all $f \in F$,

$$I(f) \ = \ \alpha \min_{\mu \in C} \int_S f(s) d\mu(s) + (1 - \alpha) \max_{\mu \in C} \int_S f(s) d\mu(s).$$

Suppose next the existence of $\alpha \in [0, 1]$ such that (α, C) provides an α-maxmin representation of $\succsim$. Axioms **B1**–**B3** follow from standard arguments while **B4** follows from the representations of $\succsim_1$ and $\succsim_2$ obtained in the first paragraph of this proof.

As for uniqueness, suppose $\alpha' \in [0, 1]$ such that (α', C) provides an α-maxmin representation of $\succsim$, and let I' denote the induced functional representing $\succsim$. For all $f \in F$, let $x = I(f)$ so that $I(f) = I(x)$ and $f \sim x$. Then, we must have $I'(f) = I'(x) = x$. Therefore, we obtain $I(f) = I'(f)$ for all $f \in F$.

Suppose finally that C is nonsingleton. Then, we may construct $f \in F$ such that

$$\min_{\mu \in C} \int_S f(s) d\mu(s) \ < \ \max_{\mu \in C} \int_S f(s) d\mu(s).$$

Moreover, the equality $I(f) = I'(f)$ implies

$$0 \ = \ (\alpha - \alpha') \cdot \left(\min_{\mu \in C} \int_S f(s) d\mu(s) - \max_{\mu \in C} \int_S f(s) d\mu(s) \right),$$

which reduces to $\alpha = \alpha'$.

Proof of Proposition 2. Suppose first **A1**–**A5** hold. Proceed as in the first paragraph of the proof of Proposition 1 to show that, for all $f, g \in F$,

$$f \ \succsim_1 \ g \iff \min_{\mu \in C} \int_S f(s) d\mu(s) \ \geq \ \min_{\mu \in C} \int_S g(s) d\mu(s),$$

and

$$f \ \succsim_2 \ g \iff \max_{\mu \in C} \int_S f(s) d\mu(s) \ \geq \ \max_{\mu \in C} \int_S g(s) d\mu(s).$$

Now, fix $f \in F$ and let $x = \min_{\mu \in C_E} \int_S f(s) d\mu(s)$. Consider any $\mu \in C$. We have

$$\int_S (f_E x)(s) d\mu(s) \ = \ \mu(E) \int_S f(s) d\mu(s|E) + \mu(E^c) x \ \geq \ \mu(E) x + \mu(E^c) x \ = \ x.$$

This shows $f_E x \succsim_1 x$. Moreover, let $\mu \in C$ be such that $x = \int_S f(s) d\mu(s|E)$. Then, we have

$$\int_S (f_E x)(s) d\mu(s) \ = \ \mu(E) \int_S f(s) d\mu(s|E) + \mu(E^c) x \ = \ \mu(E) x + \mu(E^c) x \ = \ x.$$

From there, we obtain $f_E x \sim_1 x$ and, by **A5**(iii), $f \sim_1^E x$. Fix $g \in F$ and let $y = \min_{\mu \in C_E} \int_S g(s) d\mu(s)$. By a similar argument, we obtain $g \sim_1^E y$. Since $\succsim_1^E$ is transitive and constant monotonic, i.e. **A5**(i) and **A5**(ii), it follows that, for all $f, g \in F$,

$$f \ \succsim_1^E \ g \iff \min_{\mu \in C_E} \int_S f(s) d\mu(s) \ \geq \ \min_{\mu \in C_E} \int_S g(s) d\mu(s).$$

A symmetric argument shows that, for all $f, g \in F$,

$$f \ \succsim_2^E \ g \iff \max_{\mu \in C_E} \int_S f(s) d\mu(s) \ \geq \ \max_{\mu \in C_E} \int_S g(s) d\mu(s).$$

We may then conclude the proof by applying Proposition 1 to the triple $(\succsim_1^E, \succsim_2^E, \succsim^E)$ and set C_E.

Proof of Corollary 1. Suppose C is the core of a convex capacity v on S. Then, the Choquet integral of every $f \in F$ with respect to v is equal to the minimal integral of f over C. For instance, see Proposition 3 of Schmeidler (1986). Moreover, the Choquet integral of every $f \in F$ with respect to $\overline{v}$ is equal to the maximal integral of f over C. The result readily follows from an application of Proposition 1.

As for uniqueness, suppose $\beta \in [0,1]$ is such that the functional $\beta I_1 + (1-\beta)I_2$ also represents $\succsim$. Since the representing functional is unique, we must have for all $f \in F$

$$\alpha I_1(f) + (1-\alpha)I_2(f) \;=\; \beta I_1(f) + (1-\beta)I_2(f).$$

Consider next $f \in F$ and $x \in \mathbb{R}$ such that $f \succ_2 x$ and $x \succsim_1 f$. It follows that $I_1(f) \neq I_2(f)$, and the previous formula reduces to $\alpha = \beta$.

Finally, suppose v is nonadditive. Then, by Lemma 3 and the convexity of v, there exists $E \subseteq S$ such that $v(E) + v(E^c) < 1$. We obtain $v(E) < \overline{v}(E)$. Let $x \in \mathbb{R}$ be such that $x = v(E)$ and set $f = 1_E 0$. Then, we have $f \succsim_1 x$ and $f \succ_2 x$, and the uniqueness of α follows from the previous paragraph.

Proof of Corollary 2. Suppose C is the core of a regular capacity v on S. Consider the full Bayesian update v_E of v given E. More explicitly, v_E is defined according to, for all $F \subseteq S$,

$$v_E(F) \;=\; \min_{\mu \in C_E} \mu(F).$$

Then, since v is regular, Proposition 1 of Chateauneuf et al. (2011) shows that C_E is the core of v_E. The result then follows from an application of Corollary 1 to the triple $(\succsim_1^E, \succsim_2^E, \succsim^E)$ and set C_E.

Lemma 1 *Suppose $(\succsim_1, \succsim_2)$ is standard* and also* **A6** *and* **A7** *hold. Then, there exists a closed and convex set C of probabilities on S such that C provides a maxmin representation of $\succsim_1$ and a maxmax representation of $\succsim_2$.*

Proof of Lemma 1. We first obtain two closed and convex sets C_1 and C_2 of probabilities on S such that C_1 provides a maxmin representation of $\succsim_1$ and C_2 provides a maxmax representation of $\succsim_2$. Indeed, by standard arguments, we may obtain two monotonic and continuous real-valued functionals I_1 and I_2 on F representing $\succsim_1$ and $\succsim_2$ respectively and satisfying $I_1(x) = I_2(x) = x$ for all $x \in \mathbb{R}$. See, for instance, Lemma 1 from Chateauneuf (1994).

Moreover, thanks to Item (5) in the definition of a standard* pair of binary relations, we obtain $I_1(\gamma f + x) = \gamma I_1(f) + x$ for all $f \in F$, $x \in \mathbb{R}$ and $\gamma \geq 0$. This shows in particular that I_1 is constant additive and homogenous of degree 1, and the same holds for I_2. Finally, consider $f, g \in F$ and let $x \in \mathbb{R}$ be such that $I_1(f) = I_1(g) + x$. Then, by constant additivity, $I_1(f) = I_1(g')$ and $f \sim_1 g'$ where $g' = g + x$. **A7** yields $\frac{1}{2}f + \frac{1}{2}g' \succsim_1 f$; That is, by homogeneity and constant additivity, $I_1(f+g) + x \geq 2I_1(f)$ and $I_1(f+g) \geq I_1(f) + I_1(g)$. This shows that I_1 is superadditive, and a symmetric argument shows that I_2 is subadditive. A double application of Lemma 3.5 of Gilboa and Schmeidler (1989) yields two closed and convex sets C_1 and C_2 of probabilities on S such that, for all $f \in F$,

$$I_1(f) \;=\; \min_{\mu \in C_1} \int_S f(s) d\mu(s) \quad \text{and} \quad I_1(f) \;=\; \max_{\mu \in C_2} \int_S f(s) d\mu(s).$$

From there, the maxmin and maxmax representations of $\succsim_1$ and $\succsim_2$ readily follows.

Next, we show $C_1 = C_2$. Fix $f \in F$ and let $x \in \mathbb{R}$ be such that $x = I_1(f)$. Then, we have $f \sim_1 x$. Define $g \in F$ through $g(s) = x - f(s)$ for all $s \in S$. Hence, f and g are complementary with $f + g = x$. Since $f \sim_1 x$, we have $f \sim_1 f + g$ and, by **A6**, obtain $g \sim_2 \mathbf{0}$. Put differently, we have

$$0 \;=\; I_2(g) \;=\; I_2(x - f) \;=\; x + I_2(-f) \;=\; I_1(f) + I_2(-f).$$

From there, it follows that $I_2(f) = -I_1(-f)$ for all $f \in F$. In other words, for all $f \in F$, we have

$$\max_{\mu \in C_2} \int_S f(s) d\mu(s) \;=\; -\min_{\mu \in C_1} \int_S (-f(s)) d\mu(s) \;=\; \max_{\mu \in C_1} \int_S f(s) d\mu(s).$$

A standard application of the separation theorem yields $C_1 = C_2$. Then, set $C := C_1 = C_2$ to conclude.

Proof of Proposition 3. Let C be as in Lemma 1 and suppose $\succsim$ satisfies **B1**–**B4**. Proceed as in Lemma 1 of Chateauneuf (1994) to obtain a real-valued functional I defined on F such that, for all $f, g \in F$, $f \succsim g$ if and only if $I(f) \geq I(g)$ and such that $I(x) = x$ for all $x \in \mathbb{R}$. Note that I is monotonic and constant linear. See the proof of Proposition 1. Thanks to **B4**, we can apply Lemma A.3 from Frick et al. (2022) and obtain the existence of $\alpha \in [0, 1]$ such that, for all $f \in F$,

$$I(f) \;=\; \alpha \min_{\mu \in C} \int_S f(s) d\mu(s) + (1 - \alpha) \max_{\mu \in C} \int_S f(s) d\mu(s).$$

Lemma 2 *Suppose $(\succsim_1, \succsim_2)$ is standard and also* **A6** *and* **A8** *hold. Then, there exists a convex capacity v on S such that v provides a Choquet representation of $\succsim_1$ and $\overline{v}$ provides a Choquet representation of $\succsim_2$.*

Proof of Lemma 2. We first obtain a convex capacity v_1 and a concave capacity v_2 on S that provide Choquet representations of $\succsim_1$ and $\succsim_2$ respectively. It is indeed enough to build upon the proofs of Chateauneuf (1994). Note, however, that, in the present paper, we use the classical continuity axiom **A2** which allows to construct certainty equivalents through the connexity of F and establish their uniqueness through monotonicity as captured by **A3**. Note also that our **A8** implies that each of $\succsim_1$ and $\succsim_2$ satisfies Chateauneuf's axiom A.4. of Comonotonic Independence.

Next, we show $v_2 = \overline{v}_1$. For all $f \in F$, let $I_1(f)$ and $I_2(f)$ denote the Choquet integrals of f with respect v_1 and v_2 respectively. Proceed as in the proof of Lemma 1 to show that **A6** implies $I_2(f) = -I_1(-f)$ for all $f \in F$. In other words, for all $f \in F$, we have

$$\int_S f(s) dv_2(s) \;=\; -\int_S (-f(s)) dv_1(s) \;=\; \int_S f(s) d\overline{v}_1(s).$$

Applying this to indicator functions yields $v_2 = \overline{v}_1$. It is then sufficient to set $v = v_1$ to conclude.

Lemma 3 *Consider a convex capacity v on S. Then, v is a additive if and only if $v(E) + v(E^c) = 1$ for all $E \subseteq S$.*

Proof of Lemma 3. The necessity part is obvious. Suppose now that $v(E) + v(E^c) = 1$ for all $E \subseteq S$. Fix $E, F \subseteq S$ such that $E \cap F = \emptyset$. We will show that $v(E \cup F) = v(E) + v(F)$. By convexity, we already have $v(E \cup F) \geq v(E) + v(F)$. By assumption, we have

$$v(E) = 1 - v(E^c), \; v(F) = 1 - v(F^c) \;\text{ and }\; v(E \cup F) = 1 - v(E^c \cap F^c)$$

and therefore obtain

$$v(E \cup F) - v(E) - v(F) \;=\; -v(E^c \cap F^c) - 1 + v(E^c) + v(F^c).$$

Meanwhile, the convexity of v_1 implies $v(E^c) + v(F^c) \leq v(E^c \cup F^c) + v(E^c \cap F^c)$. Since $E \cap F = \emptyset$, we have $v(E^c \cup F^c) = 1$ and obtain

$$-v(E^c \cap F^c) - 1 + v(E^c) + v(F^c) \;\leq\; 0.$$

Finally, the inequality $v(E \cup F) \leq v(E) + v(F)$ follows from the combination of the two latter formulas.

Proof of Proposition 4. Let v be as in Lemma 2 and suppose $\succsim$ satisfies **B1**–**B4**. Let I_1 and I_2 denote the Choquet integrals with respect to v and $\overline{v}$. Proceed as in Lemma 1 of Chateauneuf (1994) to obtain a real-valued functional I defined on F such that, for all $f, g \in F$, $f \succsim g$ if and only if $I(f) \geq I(g)$ and such that $I(x) = x$ for all $x \in \mathbb{R}$. Note that I is monotonic and constant linear. See the proof of Proposition 1. Thanks to **B4**, we can apply Lemma A.3 from Frick et al. (2022) and obtain the existence of $\alpha \in [0, 1]$ such that, for all $f \in F$,

$$I(f) \;=\; \alpha \int_S f(s) dv(s) + (1 - \alpha) \int_S f(s) d\overline{v}(s).$$

As for uniqueness, suppose $\beta \in [0, 1]$ is such that the functional $\beta I_1 + (1 - \beta) I_2$ also represents $\succsim$. Since the representing functional is unique, we must have for all $f \in F$

$$\alpha I_1(f) + (1 - \alpha) I_2(f) \;=\; \beta I_1(f) + (1 - \beta) I_2(f).$$

Consider next $f \in F$ and $x \in \mathbb{R}$ such that $f \succ_2 x$ and $x \succsim_1 f$. It follows that $I_1(f) \neq I_2(f)$, and the previous formula reduces to $\alpha = \beta$.

Finally, suppose v_1 is nonadditive. Then, by Lemma 3 and the convexity of v_1, there exists $E \subseteq S$ such that $v_1(E) + v_1(E^c) < 1$. We obtain $v_1(E) < \overline{v}_1(E) = v_2(E)$. Let $x \in \mathbb{R}$ be such that $x = v_1(E)$ and set $f = 1_E 0$. Then, we have $f \succsim_1 x$ and $f \succ_2 x$, and the uniqueness of α follows from the previous paragraph.

Proof of Proposition 5. Suppose $(\succsim_1, \succsim_2)$ is standard*. By standard arguments, we may obtain two monotonic and continuous real-valued functionals I_1 and I_2 on F representing $\succsim_1$ and $\succsim_2$ respectively and satisfying $I_1(x) = I_2(x) = x$ for all $x \in \mathbb{R}$. See, for instance, Lemma 1 from Chateauneuf (1994).

Moreover, thanks to Item (5) in the definition of a standard* pair of binary relations, we obtain $I_1(\gamma f + x) = \gamma I_1(f) + x$ for all $f \in F$, $x \in \mathbb{R}$ and $\gamma \geq 0$. This shows in particular that I_1 is constant additive and homogenous of degree 1, and hence constant linear. The same holds for I_2.

Suppose now **A6**. Fix $f \in F$ and let $x \in \mathbb{R}$ be such that $x = I_1(f)$. Then, we have $f \sim_1 x$. Define $g \in F$ through $g(s) = x - f(s)$ for all $s \in S$. Hence, f and g are complementary with $f + g = x$. Since $f \sim_1 x$, we have $f \sim_1 f + g$ and, by **A6**, obtain $g \sim_2 \mathbf{0}$. Put differently, we have

$$0 \;=\; I_2(g) \;=\; I_2(x - f) \;=\; x + I_2(-f) \;=\; I_1(f) + I_2(-f).$$

From there, it follows that $I_2(f) = -I_1(-f)$ for all $f \in F$. In other words, for all $f \in F$, we have

$$\max_{\mu \in C_2} \int_S f(s) d\mu(s) \;=\; -\min_{\mu \in C_1} \int_S (-f(s)) d\mu(s) \;=\; \max_{\mu \in C_1} \int_S f(s) d\mu(s).$$

Finally, suppose **A9**. Fix $f \in F$ and let $x \in \mathbb{R}$ be such that $x = I_1(f)$. Then, we have $f \sim_1 x$. By **A9**, we obtain $f \succsim_2 x$; That is, $I_2(f) \geq x$. The inequality $I_2(f) \geq I_1(f)$ follows.

Proof of Proposition 6. Suppose $\succsim$ satisfies **B1**–**B5**. Observe that, since each of $\succsim_1$ and $\succsim_2$ is monotonic in the sense of Item (4) in the definition of a standard pair of binary relations, **B4** makes sure that $\succsim$ is also monotonic in the latter sense. We can then proceed as in Lemma 1 of Chateauneuf (1994) and obtain a (unique) monotonic functional I from F to $\mathbb{R}$ representing $\succsim$ and satisfying $I(x) = x$ for all $x \in \mathbb{R}$. Moreover, by Item (5) in the definition of a standard* pair of binary relations,, I must be constant linear. (See, for instance, the proof of Proposition 5.)

By **B4**, there exists a real-valued function φ on $\Phi = \{(I_1(f), I_2(f)),\, f \in F\} \subseteq \mathbb{R}^2$ such that, for all $f \in F$,

$$I(f) \;=\; \varphi[I_1(f), I_2(f)].$$

Now, fix $f \in F$. If $I_1(f) > I(f)$, then $f \sim_1 x$ and $x \succ f$ for $x = I_1(f) \in \mathbb{R}$. By **B5**(i), we obtain $I(f) \geq I_2(f)$. If $I(f) > I_1(f)$, then $f \sim_1 x$ and $f \succ x$ for $x = I_1(f) \in \mathbb{R}$. By

B5(ii), we obtain $I_2(f) \geq I(f)$. In the two cases, $I(f)$ lies in-between $I_1(f)$ and $I_2(f)$, and this obviously still holds true if $I(f) = I_1(f)$. Overall, this shows, for all $f \in F$,

$$\min[I_1(f), I_2(f)] \;\leq\; I(f) \;\leq\; \max[I_1(f), I_2(f)].$$

Let $f \in F$ be such that $\min[I_1(f), I_2(f)] < \max[I_1(f), I_2(f)]$. Consider the case where $I_1(f) < I_2(f)$. Define $\alpha(f) \in [0,1]$ through the following formula

$$I(f) \;=\; \alpha(f)I_1(f) + (1-\alpha(f))I_2(f).$$

Then, by constant linearity, we have

$$\alpha(f) \;=\; -\frac{I(f)-I_2(f)}{I_2(f)-I_1(f)} \;=\; -I\left[\frac{f-I_2(f)}{I_2(f)-I_1(f)}\right] \;=\; -\varphi\left[\frac{I_1(f)-I_2(f)}{I_2(f)-I_1(f)}, \frac{I_2(f)-I_2(f)}{I_2(f)-I_1(f)}\right].$$

So we obtain $\alpha(f) = -\varphi(-1,0)$ which is independent of f. Set $\alpha_0 = -\varphi(-1,0)$. Consider now the case where $I_2(f) < I_1(f)$. Define $\alpha(f) \in [0,1]$ through the following formula

$$I(f) \;=\; \alpha(f)I_1(f) + (1-\alpha(f))I_2(f).$$

Then, by constant linearity, we have

$$\alpha(f) \;=\; \frac{I(f)-I_2(f)}{I_1(f)-I_2(f)} \;=\; I\left[\frac{f-I_2(f)}{I_1(f)-I_2(f)}\right] \;= \varphi\left[\frac{I_1(f)-I_2(f)}{I_1(f)-I_2(f)}, \frac{I_2(f)-I_2(f)}{I_1(f)-I_2(f)}\right].$$

So we obtain $\alpha(f) = \varphi(1,0)$ which is independent of f. Set $\alpha_1 = \varphi(1,0)$.

Suppose $\alpha_0 \leq \alpha_1$. Then, for all $f \in F$,

$$I(f) \;=\; \max_{\alpha \in [\alpha_0, \alpha_1]} \left\{ \alpha I_1(f) + (1-\alpha)I_2(f) \right\}.$$

If $\alpha_0 \geq \alpha_1$, then, for all $f \in F$,

$$I(f) \;=\; \min_{\alpha \in [\alpha_1, \alpha_0]} \left\{ \alpha I_1(f) + (1-\alpha)I_2(f) \right\}.$$

As for uniqueness, suppose $A = [\alpha_0, \alpha_1]$ and $A' = [\alpha'_0, \alpha'_1]$ provide two "max representations" of I. (The proof is similar for "min representations".) Let $f, g \in F$ and $x, y \in \mathbb{R}$ be such that $f \succ_2 x$ and $x \succsim_1 f$ while $g \succ_1 y$ and $y \succsim_2 g$. We must have $I_1(f) < I_2(f)$ and $I_1(g) > I_2(g)$. The two representations yield

$$\alpha_0 I_1(f) + (1-\alpha_0)I_2(f) = \alpha'_0 I_1(f) + (1-\alpha'_0)I_2(f)$$

and

$$\alpha_1 I_1(g) + (1-\alpha_1)I_2[(g) = \alpha'_1 I_1(g) + (1-\alpha'_1)I_2(g).$$

This is only possible if $\alpha_0 = \alpha'_0$ and $\alpha_1 = \alpha'_1$ and hence if $A = A'$.

Proof of Corollary 3. Suppose $(\succsim_1, \succsim_2)$ is standard* and satisfies **B6**. We first show that **B5** is implied. Indeed, proceed as in the proof of Proposition 6 to obtain the unique monotonic and constant linear functional I from F to $\mathbb{R}$ representing $\succsim$ and satisfying $I(x) = x$ for all $x \in \mathbb{R}$. By **B6**, we have $I_1(f) \leq I(f) \leq I_2(f)$ for all $f \in F$.

To show **B5**(i), consider $f \in F$ and $x \in \mathbb{R}$ such that $f \succsim_1 x$. Then, by **B6**, it cannot be the case that $x \succ f$. Suppose instead $f \succsim_2 x$ and $x \succ f$. Then, we have $I_2(f) \geq x$ and $x > I(f)$. Fix any $y \in \mathbb{R}$ such that $y \succsim f$. It must be that $y \geq I(f)$, and we obtain $y \geq I_1(f)$; That is, $y \succsim_1 f$. The proof of **B5**(ii) is similar.

The result then follows from an application of Proposition 5 and 6.

As for uniqueness, suppose $\alpha, \beta \in [0, 1]$ provide two representations of I. Let $f \in F$ and $x \in \mathbb{R}$ be such that $f \succ_2 x$ and $x \succsim_1 f$. We must have $I_1(f) < I_2(f)$. The two representations yield

$$\alpha I_1(f) + (1 - \alpha) I_2(f) = \beta I_1(f) + (1 - \beta) I_2(f)$$

and

$$\alpha I_1(g) + (1 - \alpha) I_2[(g) = \beta I_1(g) + (1 - \beta) I_2(g).$$

This is only possible if $\alpha = \beta$.

Proof of Corollary 4. By Proposition 6, $\succsim$ has a "max representation" or a "min representation". We prove the result in the case of a "max representation" given by $A = [\underline{\alpha}, \overline{\alpha}]$. (The proof is similar for a "min representation".) Let I be the monotonic and constant linear functional from F to $\mathbb{R}$ defined by, for all $f \in F$,

$$I(f) \; = \; \max_{\alpha \in A} \left\{ \alpha I_1(f) + (1 - \alpha) I_2(f) \right\}.$$

Define a function J from F to $\mathbb{R}$ by setting, for all $f \in F$,

$$J(f) = \frac{1}{2} \max_{\alpha \in A} \left\{ \alpha I_1(f) + (1 - \alpha) I_2(f) \right\} + \frac{1}{2} \min_{\alpha \in A} \left\{ (1 - \alpha) I_1(f) + \alpha I_2(f) \right\}.$$

Suppose first $f \in F$ is such that $I_1(f) \leq I_2(f)$. Then, we have

$$J(f) = \frac{1}{2} \left\{ \underline{\alpha} I_1(f) + (1 - \underline{\alpha}) I_2(f) \right\} + \frac{1}{2} \left\{ (1 - \underline{\alpha}) I_1(f) + \underline{\alpha} I_2(f) \right\} = \frac{1}{2} I_1(f) + \frac{1}{2} I_2(f).$$

The same conclusion also obtains when $I_1(f) \geq I_2(f)$.

Consider now two complementary $f, \overline{f} \in \mathcal{F}$ and let $x \in X$ be such that $f + \overline{f} = x$. Then, we have

$$f \sim \overline{f} \iff I(f) = I(x - f) \iff I(f) - I(-f) = x.$$

We therefore obtain

$$f \sim \overline{f} \iff \max_{\alpha \in A} \left\{ \alpha I_1(f) + (1 - \alpha) I_2(f) \right\} - \max_{\alpha \in A} \left\{ \alpha I_1(-f) + (1 - \alpha) I_2(-f) \right\} = x.$$

According to Proposition 5, we have $I_2(f) = -I_1(-f)$ and $I_1(f) = -I_2(-f)$ for all $f \in F$ and obtain from here

$$\begin{aligned} f \sim \overline{f} \quad &\Longleftrightarrow \quad \max_{\alpha \in A}\{\alpha I_1(f) + (1-\alpha)I_2(f)\} + \min_{\alpha \in A}\{\alpha I_2(f) + (1-\alpha)I_1(f)\} = x \\ &\Longleftrightarrow \quad J(f) = \frac{x}{2} \\ &\Longleftrightarrow \quad I_1(f) + I_2(f) = x. \end{aligned}$$

We now use these remarks to show that **A9** implies the additivity of J. Let $f, g \in F$ and $x, y \in X$ be such that

$$x = I_1(f) + I_2(f) \quad \text{and} \quad y = I_1(g) + I_2(g).$$

Moreover, define $\overline{f}, \overline{g} \in F$ according to $\overline{f} = x - f$ and $\overline{g} = y - g$. Then, each of the pairs $\{f, \overline{f}\}$ and $\{g, \overline{g}\}$ is made of complementary acts, and it follows from a remark above that $f \sim \overline{f}$ and $g \sim \overline{g}$. In this context, **A9** implies that $f + g \sim \overline{f} + \overline{g}$. But note that $f + g$ and $\overline{f} + \overline{g}$ are also complementary with $(f + g) + (\overline{f} + \overline{g}) = x + y$. That same remark above then yields

$$J[f+g] \;=\; \frac{x+y}{2} \;=\; J(f) + J(g).$$

In addition to being monotonic and constant linear, J is hence additive and, therefore, an expectation with respect to some probability measure μ on S. Moreover, by construction, we have, for all $f \in F$,

$$\frac{1}{2}I_1(f) + \frac{1}{2}I_2(f) \;=\; J(f) \;=\; \int_S f(s)d\mu(s).$$

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